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نموذج رقم (١٦)
اقرار والتزام بالمعايير الأخلاقية والأمانة العلمية وقوانين الجامعة الأردنية وأنظمتها وتعليماتها لطلبة الدكتوراه

أنا الطالب: رائد راجح خليل ابو عوار الرقم الجامعي: (٩٠٦٠٣٢٤)
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عنوان الاطروحة: Statistical Inference and Prediction Involving Ordered Data
(الإستدلال الإحصائي و التنبؤ بوجود الإحصائيات الرتبية)

اعلن بأنني قد التزمت بقوانين الجامعة الأردنية وأنظمتها وتعليماتها وقراراتها السارية المفعول المتعلقة باعداد اطروحات الدكتوراه عندما قمت شخصيا" باعداد اطروحتي وذلك بما ينسجم مع الأمانة العلمية وكافة المعايير الأخلاقية المتعارف عليها في كتابة الأطروحات العلمية. كما أنني أعلن بأن اطروحتي هذه غير منقولة أو مستلة من أطاريح أو كتب أو أبحاث أو أي منشورات علمية تم نشرها أو تخزينها في أي وسيلة اعلامية، وتأسيسا" على ما تقدم فانني أتحمل المسؤولية بأنواعها كافة فيما لو تبين غير ذلك بما فيه حق مجلس العمداء في الجامعة الأردنية بالغاء قرار منحي الدرجة العلمية التي حصلت عليها وسحب شهادة التخرج مني بعد صدورها دون أن يكون لي أي حق في التظلم أو الاعتراض أو الطعن بأي صورة كانت في القرار الصادر عن مجلس العمداء بهذا الصدد.

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STATISTICAL INFERENCE AND PREDICTION INVOLVING ORDERED DATA

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This Dissertation was Submitted in Partial Fulfillment of the Requirements for the Doctor of Philosophy Degree of Mathematics

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COMMITTEE DECISION

This Dissertation (Statistical Inference and Prediction Involving Ordered Data) was Successfully Defended and Approved on --------------------

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DEDICATION

This thesis is dedication to :

my parents,

my wife,

my brothers and sisters,

my sweet son and daughters: Ahmad, Leen and Rama.

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STATISTICAL INFERENCE AND PREDICTION INVOLVING ORDERED

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ABSTRACT

In this thesis, we consider the problem of estimating the shape and scale parameters of Weibull model and predicting the missing and future data based on progressive type II censored and record samples which are coming from the Weibull model. Maximum likelihood and Bayesian approaches are used to estimate the scale and shape parameters. One-sample and two-sample prediction problems are also considered . The Gibbs sampler method is used to draw Markov Chain Monte Carlo (MCMC) samples and it has been used to compute the Bayes estimates and also to compute the point predictors of the missing and future data. Monte carlo simulations are performed to study the behavior of the proposed methods, and two real examples are presented for illustrative purposes.

CHAPTER 1

Introduction

1.1 The Weibull Model

One of the most famous distributions is the Weibull distribution since it is an important distribution in analyzing skewed data and it's an appropriate model in reliability and life-testing problems such as time to failure or life length of a component or a product. The Weibull distribution was proposed by Waloddi Weibull in 1937 for estimating machinery lifetime, see for example Weibull(1961). Extensive work has been done since then. A detailed discussion of the Weibull distribution has been provided by Johnson et al. (1995).

The two-parameter Weibull probability density function (PDF) is given by

$$f(x|\alpha,\lambda)=\begin{cases}\alpha\lambda x^{\alpha-1}e^{-\lambda x^{\alpha}} & \text{if } x>0,\\ 0 & \text{if } x\le 0.\end{cases} \tag{1.1}$$

Here $\alpha>0$ and $\lambda>0$ are the shape and scale parameters. The Weibull distribution with the shape and scale parameters α and λ is denoted by WE(α,λ).

The cumulative distribution function (CDF) of the Weibull distribution is given by

$$F(x|\alpha,\lambda)=1-e^{-\lambda x^{\alpha}},\ x>0,\ \alpha,\lambda>0. \tag{1.2}$$

1.2 Maximum Likelihood Estimator (MLE)

Let $Y_1, Y_2, ..., Y_n$ be a random sample from $f(y|\theta)$. The maximum likelihood estimator of θ is the value $\widehat{\theta}$ which maximizes the likelihood function

$$L(data|\theta) = f(y_1, y_2, ..., y_n|\theta) = \prod_{i=1}^{n} f(y_i|\theta),$$

where f is the PDF of the underlying distribution.

1.3 Bayesian Approach

One of the most popular technique in analyzing a wide variety of models is the Bayesian approach. In a Bayesian approach, θ is considered to be random variable with a distribution (called prior distribution). Bayesian inference is based on the observed data $\underset{\sim}{y} = \{y_1, ..., y_n\}$ and the prior distribution of θ.

1.3.1 Bayesian Estimation

Suppose we are interested in estimating θ from the data $\underset{\sim}{y} = \{y_1, ..., y_n\}$ by using a statistical model described by a density $f(\underset{\sim}{y}|\theta)$. The following steps describe the essential elements of Bayesian estimation :

1. A probability distribution for θ is formulated as $\pi(\theta)$, which is the prior distribution, or just the prior.
2. Given the observed data $\underset{\sim}{y}$, we choose a statistical model $f(\underset{\sim}{y}|\theta)$ to describe the distribution of $\underset{\sim}{y}$ given θ.
3. We update our belief about θ using the prior distribution to conclude the posterior distribution $\pi(\theta|\underset{\sim}{y})$. This step is carried out as follows

$$\pi(\theta|\underset{\sim}{y}) = \frac{f(\underset{\sim}{y}, \theta)}{f(\underset{\sim}{y})} = \frac{f(\underset{\sim}{y}|\theta)\pi(\theta)}{f(\underset{\sim}{y})} = \frac{f(\underset{\sim}{y}|\theta)\pi(\theta)}{\int f(\underset{\sim}{y}|\theta)\pi(\theta)\, d\theta}.$$

The Bayes estimation depends on the prior distribution(s) of the parameter(s) of a statistical model and the loss function used. Hence it is necessary to define the prior distribution and determine the most important of prior's types as well as the most important forms of the loss function that is used in the Bayesian estimation.

[1] Prior Distributions and their Assumptions

There are two kinds of prior distributions, namely the non-informative prior and informative prior distributions.
Non-informative prior distributions are associated with situations where the prior distributions have no population basis. They are used when we have little prior information, and hence the prior distributions play a minimal role in the posterior distribution. The informative prior distribution has its own parameters, which are called hyperparameters. The conjugate distribution is commonly used in the general Bayesian approach as informative prior. The prior distribution is said to be a conjugate prior for a family of distributions if the prior and posterior distributions are from the same family.
In order to perform a Bayesian estimation of the parameters α and λ of Weibull distribution, the prior distribution of α and λ must be specified either when the shape parameter α known or unknown.
When the shape parameter α is known, the prior on λ is the conjugate gamma prior. The prior distribution Gamma(a,b) of the scale parameter λ is given by

$$\pi_1(\lambda|a,b) = \begin{cases} \frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda} & \text{if } \lambda > 0, \\ 0 & \text{if } \lambda \leq 0. \end{cases} \tag{1.3}$$

Here the hyperparameters $a > 0$ and $b > 0$, $\Gamma(a)$ is the gamma function which is defined by

$$\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}\,dx. \tag{1.4}$$

On the other hand, when both parameters α and λ are unknown, it is assumed that λ has the same gamma prior as (1.3). For more details, see for example Berger and Sun (1993) or Kundu (2008). It is well known that the prior on α, $\pi_2(.)$, has no specific form. It is assumed that $\pi_2(.)$ here such that the support of $\pi_2(\alpha)$ is $(0,\infty)$ and its PDF is log-concave. Note that many density functions are log-concave, for example normal density. When the shape parameter is greater than one, the gamma or Weibull also have log-concave density functions.
To simplify the computations of the Bayes estimators and their corresponding credible intervals, in chapter 5, we need to assume specific form of $\pi_2(.)$, and will depend on some hyperparameters.

[2] Loss Functions

In statistics, a loss function represents the loss associated with an error in estimation. Different loss functions are considered for estimating some parameter θ using Bayesian approach. The first is the square error loss function and is given by

$$L_1(\theta, \delta) = (\theta - \delta)^2.$$

The most common loss function is L_1 which has been considered in the Bayesian estimation and prediction. Under the loss function L_1, the Bayes estimator is the posterior mean, *i.e.*

$$\widehat{\theta}_{B1} = \text{Mean of the posterior distribution} = E_{posterior}(\theta|\underset{\sim}{Y}).$$

The absolute loss function is defined by

$$L_2(\theta, \delta) = |\theta - \delta|.$$

Our second loss function L_2, is the symmetric loss function. In this case the Bayes estimator can be obtained as the posterior median, *i.e.*

$$\widehat{\theta}_{B2} = \text{Median of the posterior distribution} = Med_{posterior}(\theta|\underset{\sim}{Y}).$$

Both the square error loss function and the absolute error loss function are symmetric. Varian(1975) proposed asymmetric flexible linear-exponential (LINEX) loss function as follows

$$L_3(\theta, \delta) = \left(\frac{\delta}{\theta}\right)^{a^*} - a^* \ln\left(\frac{\delta}{\theta}\right) - 1, \; a^* \neq 0.$$

In this case the Bayes estimator of θ will be

$$\widehat{\theta}_{B3} = \left[E_{posterior}(\theta^{-a^*}|\underset{\sim}{Y})\right]^{-\frac{1}{a^*}}.$$

1.3.2 Credible Interval

The interval (C_L, C_U) is said to be a $(1-\beta)100\%$ credible interval for θ if

$$P(C_L < \theta < C_U|data) = \int_{C_L}^{C_U} \pi(\theta|data)\, d\theta = 1-\beta,\ 0<\beta<1.$$

Notice that, the credible interval for some parameter θ depends on the posterior distribution.

1.3.3 Bayesian Prediction

In this section, we are mainly interested in the posterior prediction density of unobserved data based on observed ones. Let $\underset{\sim}{y}$ be the observed data, θ be the parameter and y^{Pred} be the unobserved value. The posterior predictive distribution of y^{Pred} is given by

$$f(y^{Pred}|\underset{\sim}{y}) = \int f(y^{Pred}|\theta)\pi(\theta|\underset{\sim}{y})\, d\theta, \tag{1.5}$$

where $\pi(\theta|\underset{\sim}{y})$ is the posterior distribution of θ given data.

For different applications of the prediction problems, one may refer to Kaminsky and Rhodin (1985), Al-Hussaini (1999), and Madi and Raqab (2004). There are mainly two important prediction problems known as (i) one-sample prediction problem, (ii) two-sample prediction problem.

(i) One-sample prediction problem :
Let $T_1 < ... < T_r$ be the observed order statistics known as the informative sample and $T_{r+1} < ... < T_n$ be the unobserved future order statistics from the same sample, which is yet to be observed. A one-sample prediction problem involves the prediction of the future order statistics $T_{(r+k)}$; for $1 \le k \le n-r$.

(ii) Two-sample prediction problem :
In this case, let $T_1 < ... < T_r$ be the same as in (i) and $Y_1 < ... < Y_m$ be the future order statistics from another independent sample of size m of the same population. A two-sample prediction problem involves the prediction of the future order statistics Y_k; for $1 \le k \le m$.

1.4 Gibbs Sampling Method and MCMC Method

The Markov Chain Monte Carlo (MCMC) method is a general simulation method for sampling from posterior distributions and computing posterior quantities of interest. Computing posterior quantities, for example the posterior mean, could be obtained via integration. But, often times the integration does not have a closed form in most cases. We use Monte Carlo integration to approximate the integration with no closed form by using the Markov chain samples. In simulation we approximate the integration by

$$\int_S g(\theta)p(\theta|data)\,d\theta \cong \frac{1}{N}\sum_{t=1}^{N} g(\theta^t),$$

where g(.) is a function of interest, $p(\theta|data)$ is the posterior distribution of θ and θ^t are MCMC samples from $p(\theta)$ on its support S and N is the number of desired samples. Let $\underset{\sim}{\theta} = (\theta_1, ..., \theta_r)$ be the parameter vector of a certain statistical model, the posterior distribution of $\underset{\sim}{\theta}$ given the data is denoted by $\pi(\underset{\sim}{\theta}|data)$.
The Gibbs sampler is a Markov chain algorithm to draw samples from the posterior distribution $\pi(\underset{\sim}{\theta}|data)$ which works as follows :

- **Step 1**
  Randomly choose an arbitrary initial value of $\underset{\sim}{\theta} = (\theta_1, ..., \theta_r)$ as $\underset{\sim}{\theta}^{(0)} = (\theta_1^{(0)}, \theta_2^{(0)} ...., \theta_r^{(0)})$.
  Set $t = 1$.

- **Step 2**
  Generate each component of $\underset{\sim}{\theta}$ as follows :
  - draw $\theta_1^{(t)}$ from $\pi(\theta_1|\theta_2^{(t-1)}, \theta_3^{(t-1)}, ..., \theta_r^{(t-1)}, data)$,

  - draw $\theta_2^{(t)}$ from $\pi(\theta_2|\theta_1^{(t)}, \theta_3^{(t-1)}, ..., \theta_r^{(t-1)}, data)$,

  - draw $\theta_3^{(t)}$ from $\pi(\theta_3|\theta_1^{(t)}, \theta_2^{(t)}, \theta_4^{(t-1)}, ..., \theta_r^{(t-1)}, data)$,

  ...

  - draw $\theta_r^{(t)}$ from $\pi(\theta_r|\theta_1^{(t)}, \theta_2^{(t)}, \theta_3^{(t)}, ..., \theta_{(r-1)}^{(t)}, data)$.

- **Step 3**
  Set t=t+1. If $t \leq N$, return to step 2. Otherwise stop.

1.5 Type II Censored Data

A sample of size n is said to be type II censored when only its m smallest lifetimes are observed ($1 \leq m < n$). Experiments involving type II censoring are often used in lifetime testing. For example, a total of n items is placed on test, but instead of continuing until all n items have failed, the test is terminated at the time the m^{th} item fails. The number of observations m is usually decided before the data are collected. Such tests can save time and money, since it could take a very long time for all the items to fail in some instances.

The resulting data consist of the m smallest lifetimes $t_1 < t_2 < ... < t_m$ out of a random sample of n iid lifetimes presumed to have a continuous distribution with PDF $f(t;\theta)$ and CDF $F(t;\theta)$.

The likelihood function of this sample is given by

$$L(\underset{\sim}{\theta}|data) = \frac{n!}{(n-m)!}\prod_{i=1}^{m} f(t_i;\theta).[1 - F(t_m;\theta)]^{n-m},$$

where $\underset{\sim}{\theta}$ is the parameters vector of the density $f(t;\theta)$.

1.6 Progressive Type II Censored Data

A generalization of type II censoring is the progressive type II censoring. It can be described as follows :

Suppose that n units are placed in a life-testing experiment and only $m(< n)$ are observed until failure. The censoring occurs progressively in m stages. These m stages offer failure times of the m observed units. At the time of the first failure (the first stage) $X_{1:m:n}$, r_1 of the $n-1$ surviving units are randomly removed (censored) from the experiment. Similarly, at the time of the second failure (the second stage) $X_{2:m:n}$, r_2 of the $n-2-r_1$ surviving units are randomly removed (censored) from the experiment.

Finally, at the time of the m^{th} failure (the m^{th} stage) $X_{m:m:n}$, all the remaining $r_m = n-m-(r_1+r_2+...+r_{m-1})$ surviving units are removed from the experiment. We will refer to this as progressive type II censoring scheme $(r_1, r_2, ..., r_m)$. Notice that this scheme includes the type II censoring scheme $(r_1 = r_2 = ... = r_{m-1} = 0, r_m = n-m)$.

Suppose that $X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n}$ are a progressively type II censored sample of size m from a sample of size n with progressive censoring $(r_1, r_2, ..., r_m)$. We assume that $X_{i:m:n}$; $i = 1, 2, ..., m$ are iid with PDF $f(.)$ and CDF $F(.)$, the likelihood function of this sample is given by [see Balakrishnan and Aggarwala (2000)]

$$L(\underset{\sim}{\theta}|data) = C \prod_{i=1}^{m} f(x_{i:m:n}).[1 - F(x_{i:m:n})]^{r_i}, -\infty < x_{1:m:n} < ... < x_{m:m:n} < \infty \tag{1.6}$$

where $C = n(n - r_1 - 1)(n - 2 - r_1 - r_2)...(n - m + 1 - r_1 - r_2 - ... - r_{m-1})$ and $\underset{\sim}{\theta}$ is the parameters vector of the density $f(.)$.

Progressive type II censored sampling is an important method of obtaining data in lifetime studies. For more details on progressively censored samples, see for instance, Aggarwala (1996) or Balakrishnan and Aggarwala (2000).

When data are obtained by progressively censoring, several inference and prediction problems for various models have appeared in the literature. An interesting real application of progressively type II censored data has been carried out by Montanari and Cacciari (1988) by studying the wear of an insulated cable having a Weibull lifetime model.

Balakrishnan and Aggarwala (2000) have developed algorithm to simulate general progressively type II censored samples from the uniform or any other continuous distributions.

1.7 Record Data

Let $X_1, X_2, ...$ be a sequence of independent and identically random variables with PDF $f(x)$ and CDF $F(x)$. Let $Y_n = max\{(X_1, X_2, ..., X_n), n \geq 1\}$, we say that X_j is an upper record value and denoted by $X_{U(j)}$ if $Y_j > Y_{j-1}$, $j > 1$.

The indices at which the upper record values occur are called record times $\{U(n), n > 0\}$.

Many properties of the records sequence can be expressed in terms of the cummulative hazard function

$$H(x) = \int_{-\infty}^{x} h(t)\, dt = -\ln(1 - F(x)), \tag{1.7}$$

where $h(t)$ is the hazard function.

The marginal PDF of $X_{U(n)}$ [see Arnold et al. (1998), and Ahsanullah (2009)] is given by

$$f_n(x) = \frac{[H(x)]^{n-1}}{(n-1)!} f(x). \tag{1.8}$$

The conditional distribution of $X_{U(j)}$ given $X_{U(i)} = x_i$ [see Arnold et al. (1998), and Ahsanullah (2009)] is similarly given by

$$f_{X_{U(j)}|X_{U(i)}}(x_j|x_i) = \frac{[H(x_j) - H(x_i)]^{j-i-1}}{(j-i-1)!} \frac{f(x_j)}{1 - F(x_i)}, for - \infty < x_i < x_j < \infty. \tag{1.9}$$

Record values arise naturally in many real life applications involving data relating to weather, sports, economics and life-tests. For an elaborate treatment on records and their applications, one may refer to books by Arnold et al. (1998), Nevzorov (2000) and Gulati and Padgett (2003).

1.8 Problem Statement

In this research work, we study different methods of estimation and prediction involving progressive type II censored data and record data. These methods involve the classical method (the MLE method) as well as the Bayesian approach.

The organization of this thesis is as follows :

In chapter 2, we review and describe the literature relevant to the topic under study in this thesis.

In chapter 3, and based on progressive type II censored sample from the Weibull distribution WE(α, λ), the maximum likelihood method is used to estimate the shape parameter α and scale parameter λ. When the shape parameter α is known and unknown Bayesian approaches are used to estimate α and λ, or some function of α and λ, say $\theta = g(\alpha, \lambda)$, under different loss functions described in subsection [1.3.1], the symmetric credible intervals are also established. When the shape parameter is unknown, the Bayes estimators of α and λ can't be obtained in closed forms.

We use Gibbs sampling procedure to draw MCMC samples and has been used to compute the Bayes estimators and also to construct symmetric credible intervals for α and λ. One-sample and two-sample prediction problems are used to predict the missing and future data based on observed sample.

In chapter 4, we use record sample from WE(α, λ) to estimate the shape and scale parameters as well as predict of future observations.

Numerical study is demonstrated in chapter 5 based on simulation data.

CHAPTER 2

Literature Review

In this chapter we describe the literature relevant to the topic under study in this thesis.
Raqab et al. (2007) obtained the maximum likelihood estimator and Bayes estimators for the parameters of the Pareto model based on the record data. Also, they considered the problem of prediction for the future record values based on some observed record values.

Soliman and Al-Aboud (2008) obtained the maximum likelihood estimator and Bayes estimators for the parameters of the Rayleigh distribution based on the record data. Also, they considered the problem of prediction for the future record values based on some observed record values.

Madi and Raqab (2009) studied the Bayesian estimation of the parameters as well as prediction of the unobserved failure times from the generalized exponential (GE) distribution, based on progressively censoring sample data. They used the Gibbs sampler for predicting times to failure of units in multiple stages.

Kundu and Howlader (2010) described the Bayesian inference and prediction of the inverse Weibull distribution for type II censored data. They obtained the Bayes estimator of the unknown parameter based on the square error loss function. They used tha Gibbs sampler to draw MCMC samples for estimating the two unknown parameters of the inverse Weibull distribution.

Mousa and Al-Sagheer (2005) used the two-sample prediction problem to predict the kth order statistics in the future progressive sample based on observed progressive sample from Rayleigh distribution.
Kundu and Raqab (2011) described the Bayesian inference and prediction of the two parameter Weibull distribution when the data are type II censored. They used the Gibbs sampling procedure to draw MCMC samples to compute the Bayes estimators

and construct symmetric credible intervals. Their work is summarized as follows :

Suppose n individual units are put on a test and let us denote the lifetimes of the n units by $T_1, T_2, ..., T_n$. It is assumed that T_i's are independent and identically distributed random variables with PDF(1.1), *i.e.*

$$f(t|\alpha,\lambda) = \begin{cases} \alpha\lambda t^{\alpha-1}e^{-\lambda t^{\alpha}} & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

The integer $m < n$ is pre-fixed and the experiment stops as soon as we observe the m^{th} failure. We denote the first m failure points as $t_1 < t_2 < ... < t_m$. The problem was to estimate and construct credible interval of some function of α and λ, say $g(\alpha,\lambda) = \theta$, or just only of α or λ.
They considered the Bayesian estimation in two cases :

Case(1) Shape Parameter (α) Known.

Based on the type II censored sample $t_1 < t_2 < ... < t_m$, and when λ has the prior distribution as mentioned in subsection [1.3.1] of chapter 1, Eq.(1.3), the posterior density function of λ is well known to be
$Gamma(a+m,\ b+\sum\limits_{i=1}^{m} t_i^{\alpha} + (n-m)t_m^{\alpha})$, *i.e.*

$$\pi(\lambda|\alpha, data) = \frac{\left(b+\sum\limits_{i=1}^{m} t_i^{\alpha} + (n-m)t_m^{\alpha}\right)^{a+m}}{\Gamma(a+m)}\ \lambda^{a+m-1}\ e^{-\lambda(b+\sum\limits_{i=1}^{m} t_i^{\alpha}+(n-m)t_m^{\alpha})}. \tag{2.1}$$

Therefore, the Bayes estimator of λ under the loss function L_1 is the posterior mean and that is

$$\hat{\lambda}_1 = \frac{a+m}{b+\sum\limits_{i=1}^{m} t_i^{\alpha} + (n-m)t_m^{\alpha}}.$$

The Bayes estimator of λ under the loss function L_2 is the posterior median and that is

$$\hat{\lambda}_2 = \frac{m+c_1}{b+\sum\limits_{i=1}^{m} t_i^{\alpha} + (n-m)t_m^{\alpha}} + O(m^{-3}),$$

here $c_1 = a - \frac{1}{3} + \frac{8}{405m}$, [see Ren et al. (2006)].
The Bayes estimator of λ under the loss function L_3 is

$$\hat{\lambda}_3 = \frac{m + c_2}{b + \sum_{i=1}^{m} t_i^{\alpha} + (n-m)t_m^{\alpha}} + O(m^{-3}),$$

here $c_2 = a - \frac{(a^*+1)}{2} - \frac{(a^*-1)}{24m}$, [see Lemma 5 of Ren et al. (2006)].
Because, the posterior distribution of λ follows gamma, a credible interval of λ can be easily obtained. Moreover, if $a+m$ is a positive integer, then the chi-square table values can be used for constructing credible intervals.

Case(2) Shape Parameter (α) Unknown.

It is assumed that α and λ have the joint prior as described in subsection [1.3.1] of chapter 1. The posterior distribution of α and λ given the data is denoted by $\pi(\alpha, \lambda|data)$.
The Bayes estimator of $\theta = g(\alpha, \lambda)$ under the loss function L_1 is obtained by

$$\hat{\theta}_{B_1} = E_{posterior}(\theta|data) = \int_0^{\infty}\int_0^{\infty} \theta\pi(\alpha, \lambda|data)\, d\alpha\, d\lambda.$$

The Bayes estimator of $\theta = g(\alpha, \lambda)$ under the loss function L_2 is given by

$$\hat{\theta}_{B_2} = Med_{posterior}(\theta|data).$$

The Bayes estimator of $\theta = g(\alpha, \lambda)$ under the loss function L_3 is obtained by

$$\hat{\theta}_{B_3} = \left[E_{posterior}(\theta^{-a^*}|data)\right]^{-\frac{1}{a^*}} = \left[\int_0^{\infty}\int_0^{\infty} \theta^{-a^*}\pi(\alpha, \lambda|data)\, d\alpha\, d\lambda\right]^{-\frac{1}{a^*}}.$$

It is clear that in their work even if $\pi_2(\alpha)$ has a specific form, the Bayes estimators with respect to different loss functions may not be obtained in explicit forms. So they used Gibbs sampling technique to draw MCMC samples from the posterior distribution of α and λ, $\pi(\alpha, \lambda|data)$, and hence the Bayes estimators can be obtained. The Gibbs sampler method, as mentioned in section [4] of chapter 1, needs the conditional distributions $\pi(\lambda|\alpha, data)$ and $\pi(\alpha|data)$.
The conditional distribution $\pi(\lambda|\alpha, data)$ is in Eq.(2.1). Kundu and Raqab (2011) state the following theorem for this conditional distribution $\pi(\alpha|data)$:

Theorem

The conditional PDF of α given data is given by

$$\pi(\alpha|data) \propto \pi_2(\alpha)\,\alpha^m \prod_{i=1}^{m} t_i^{\alpha-1} \times \frac{1}{\left(b + \sum_{i=1}^{m} t_i^{\alpha} + (n-m)t_m^{\alpha}\right)^{a+m}},$$

and it is log-concave.

To obtain the Bayes estimator under the three loss functions, Kundu and Raqab (2011) proposed the following algorithm of Gibbs sampler to draw MCMC samples

- Step 1 Generate α from $\pi(\alpha|data)$ using the general methodology of Devroye (1984).
- Step 2 For a given α, generate λ from $\pi(\lambda|\alpha, data)$.
- Step 3 Repeat steps 1 and 2 N times and obtain MCMC samples $\{(\alpha_i, \lambda_i); i = 1, 2, ..., N\}$ and $\theta_i = g(\alpha_i, \lambda_i)$.

Based on MCMC samples $\{(\alpha_i, \lambda_i); i = 1, 2, ..., N\}$, Kundu and Raqab (2011) obtained the Bayes estimators under the different loss functions and also constructed credible intervals, by using the method of Monte Carlo described in section [4] of chapter 1. As for the prediction problem, extensive work can be found in the literature.

Based on the MCMC samples obtained in step 3 of the previous algorithm, Kundu and Raqab (2011) have obtained the lower and upper bounds of a $(1-\beta)100\%$, $0 < \beta < 1$, and predictive intervals of future observations based on the past sample when the data are type II censored.

Smith (1997) investigated the asymptotic property of the predictive inference of Bayes and frequentist procedures for a class of parametric family.

Al-Hussaini (1999) also considered the Bayesian prediction problem for a large class of lifetime distributions.

CHAPTER 3

Statistical Inference Based on Progressively Type II Censored Data from Weibull Model

3.1 Maximum Likelihood Estimation

In this section, we derive the maximum likelihood estimator of the parameters α and λ of the Weibull model, based on the progressive type II censored data.
Suppose that $\underset{\sim}{X} = (X_{1:m:n}\,,\, X_{2:m:n}\,,\, ...\,,\, X_{m:m:n})$ is a progressively type II censored sample of size m from a sample of size n drawn from a Weibull distribution, with progressive censoring scheme $(r_1\,,\, r_2\,,\, ...\,,\, r_m)$. Suppose that $X_{i:m:n}\,;\; i = 1, 2, ..., m$ are iid random variables with PDF (1.1) and CDF (1.2) being defined in chapter 1. Based on Eq.(1.6), the likelihood function of this sample is given by

$$L(\alpha, \lambda|\underset{\sim}{X}) = C \prod_{i=1}^{m} f(x_{i:m:n}|\alpha, \lambda)\, [1 - F(x_{i:m:n}|\alpha, \lambda)]^{r_i}\,, \tag{3.1}$$

where $C = n(n - r_1 - 1)(n - 2 - r_1 - r_2) \ldots (n - m + 1 - r_1 - r_2 - \, \ldots \, - r_{m-1})$.
From (1.1), (1.2) and (3.1), we write the likelihood function as follows

$$L(\alpha, \lambda|\underset{\sim}{X}) = C\alpha^m \lambda^m \left(\prod_{i=1}^{m} x_{i:m:n}^{\alpha-1} \right) e^{-\lambda \sum\limits_{i=1}^{m} (1+r_i) x_{i:m:n}^{\alpha}}. \tag{3.2}$$

The natural logarithm of the likelihood function, Eq.(3.2), is

$$\ln L(\alpha, \lambda|\underset{\sim}{X}) = \ln C + m \ln \alpha + m \ln \lambda + (\alpha - 1) \sum_{i=1}^{m} \ln x_{i:m:n} - \lambda \sum_{i=1}^{m} (1 + r_i) x_{i:m:n}^{\alpha}. \tag{3.3}$$

By differentiating Eq.(3.3), with respect to λ and equating the resulting terms to zero, we obtain the estimating equation

$$\frac{\partial}{\partial\lambda}\ln L = \frac{m}{\lambda} - \sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha} = 0,$$

which gives

$$\lambda = \frac{m}{\sum\limits_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha}}. \tag{3.4}$$

If we differentiate Eq.(3.3), with respect to α and equating the resulting terms to zero, we obtain the following estimating equation

$$\frac{\partial}{\partial\alpha}\ln L = \frac{m}{\alpha} + \sum_{i=1}^{m}\ln x_{i:m:n} - \lambda\sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha}\ln x_{i:m:n} = 0. \tag{3.5}$$

By substituting Eq.(3.4) into Eq.(3.5), we obtain

$$\frac{m}{\alpha} + \sum_{i=1}^{m}\ln x_{i:m:n} - \frac{m}{\sum\limits_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha}} \times \sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha}\ln x_{i:m:n} = 0. \tag{3.6}$$

If we use a suitable numerical method, the solution of a non-linear Eq.(3.6) will be the maximum likelihood estimate of α, and by substituting the value of α into Eq.(3.4), we obtain the maximum likelihood estimate of λ.

3.2 Bayes Estimation and Credible Intervals

In this section, we estimate the unknown scale parameter λ and it's corresponding credible interval when the shape parameter α is known. When both shape parameter α and scale parameter λ are unknown, we use the Gibbs sampling method to estimate the two parameters α and λ, under different loss functions and with respect to the prior(s) described in subsection [1.3.1] of chapter 1. Also the credible intervals of α and λ are considered.

3.2.1 Shape Parameter Known

Based on a progressive type II censored data $\underset{\sim}{X}$ described in section [3.1], and by combining the likelihood function, Eq.(3.2), and the prior density of λ, Eq.(1.3), we obtain the following theorem

Theorem 1

The conditional PDF of λ given α and $\underset{\sim}{X}$ is $Gamma(a+m,\ b+\sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha})$. That is, the posterior density of λ given α and $\underset{\sim}{X}$ is of the form

$$\pi_1(\lambda|\alpha,\underset{\sim}{X}) = \frac{\left(b+\sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha}\right)^{a+m}}{\Gamma(a+m)}\,\lambda^{a+m-1}\,e^{-\lambda(b+\sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha})}. \tag{3.7}$$

Proof

From Eq.(3.2) and Eq.(1.3), we immediately have

$$\begin{aligned}\pi_1(\lambda|\alpha,\underset{\sim}{X}) &\propto L(\alpha,\lambda)\,\pi(\lambda|a,b)\\ &\propto \lambda^{m}\,e^{-\lambda\sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha}}\,\lambda^{a-1}e^{-\lambda b}\\ &= \lambda^{a+m-1}\,e^{-\lambda(b+\sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha})}.\end{aligned}$$

It follows that $\lambda|\alpha,\underset{\sim}{X} \sim Gamma(a+m,\ b+\sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha})$.

Under the square error loss function L_1, the Bayes estimator $\hat{\lambda}_{B_1}$ of λ is given by

$$\hat{\lambda}_{B_1} = E_{posterior}(\lambda|\alpha,\underset{\sim}{X}) = \int_0^{\infty}\lambda\pi(\lambda|\alpha,\underset{\sim}{X})\,d\lambda = \frac{a+m}{b+\sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha}}. \tag{3.8}$$

The Bayes estimator $\hat{\lambda}_{B_2}$ of λ with respect the loss function L_2 is the median of the posterior density function. In this case we do not have explicit expression of the median. By using Lemma 1 of Ren et al. (2006), the Bayes estimator will be

$$\hat{\lambda}_{B_2} = Med_{posterior}(\lambda|\alpha,\underset{\sim}{X}) = \frac{m+c_1}{b+\sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha}} + O(m^{-3}), \tag{3.9}$$

where $c_1 = a - \frac{1}{3} + \frac{8}{405m}$.

The Bayes estimator $\hat{\lambda}_{B3}$ of λ under the LINEX loss function L_3 with $a^* \neq 0$, can be seen as

$$\hat{\lambda}_{B3} = \left[E_{posterior}(\lambda^{-a^*}|\alpha, \underset{\sim}{X})\right]^{-\frac{1}{a^*}}.$$

Now, by using Lemma 5 of Ren et al. (2006), it can be seen that

$$\hat{\lambda}_{B_3} = \frac{m + c_2}{b + \sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha}} + O(m^{-3}), \tag{3.10}$$

where $c_2 = a - \frac{(a^*+1)}{2} - \frac{(a^*-1)}{24m}$.
Since, the posterior distribution of λ follows gamma distribution, a credible interval of λ can be obtained as follows :
The $(1-\beta)100\%$ credible interval of λ, (C_L, C_U), satisfies the following two conditions

$$P(C_L < \lambda < \infty) = 1 - \frac{\beta}{2}, \tag{3.11}$$

$$P(C_U < \lambda < \infty) = \frac{\beta}{2}. \tag{3.12}$$

Now from Eq.(3.11), we have

$$\int_{C_L}^{\infty} \frac{\left(b + \sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha}\right)^{a+m}}{\Gamma(a+m)} \lambda^{a+m-1} e^{-\lambda\left(b+\sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha}\right)} d\lambda = 1 - \frac{\beta}{2}.$$

By making the transformation $u = \lambda(b + \sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha})$, we immediately obtain

$$\int_{\left(b+\sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha}\right)C_L}^{\infty} \frac{u^{a+m-1}\, e^{-u}}{\Gamma(a+m)} du = 1 - \frac{\beta}{2},$$

which is equivalently to

$$\int_{\left(b+\sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha}\right)C_L}^{\infty} u^{a+m-1}\, e^{-u}\, du = \left(1 - \frac{\beta}{2}\right)\Gamma(a+m).$$

By using the incomplete gamma function, which is defined as

$$\Gamma(a, c) = \int_{c}^{\infty} x^{a-1} e^{-x}\, dx,\ a > 0,\ c > 0, \tag{3.13}$$

we immediately obtain

$$\Gamma\left(a+m,\ (b+\sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha})C_L\right)=\left(1-\frac{\beta}{2}\right)\Gamma(a+m). \tag{3.14}$$

Similarly from Eq.(3.12), we obtain

$$\Gamma\left(a+m,\ (b+\sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha}C_U\right)=\frac{\beta}{2}\Gamma(a+m). \tag{3.15}$$

By using a suitable numerical method, we obtain the lower and upper credible interval C_L and C_U by solving the equations (3.14) and (3.15), with respect to C_L and C_U, respectively.

In particular, if a is positive integer, then the chi-square table values can be used for constructing credible interval for λ as follows :

Since λ has $Gamma(a+m,\ b+\sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha})$, then a pivotal statistic $Q=2\lambda(b+\sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha})$ has $\chi^2_{2(a+m)}$. Hence, the $(1-\beta)100\%$ credible interval for λ is given by

$$\frac{\chi^2_{(1-\frac{\beta}{2},\,2(a+m))}}{2(b+\sum_{i=1}^{m}x_{i:m:n}^{\alpha}(1+r_i))}<\lambda<\frac{\chi^2_{(\frac{\beta}{2},\,2(a+m))}}{2(b+\sum_{i=1}^{m}x_{i:m:n}^{\alpha}(1+r_i))},$$

where $\chi^2_{(\beta,r)}$ is the $\beta 100th$ upper percentile of chi-square with r degrees of freedom.

3.2.2 Shape Parameter Unknown

In this subsection, we describe the procedure to obtain different Bayes estimators of α and λ, or in general of $\theta = g(\alpha, \lambda)$, under different loss functions described in subsection [1.3.1] of chapter 1, when both parameters α and λ are unknown. It is assumed that α and λ have the joint prior described in subsection [1.3.1] of chapter 1, based on the prior distributions $\pi_1(\lambda|a,b)$ and $\pi_2(\alpha)$, the posterior distribution of α and λ is given by

$$\pi(\alpha, \lambda|\underset{\sim}{X}) = \frac{L(\alpha, \lambda|\underset{\sim}{X}) \,.\, \pi_1(\lambda|\alpha, a, b)\pi_2(\alpha)}{\int_0^\infty \int_0^\infty L(\alpha, \lambda|\underset{\sim}{X}) \,.\, \pi_1(\lambda|\alpha, a, b)\pi_2(\alpha)\, d\alpha\, d\lambda}. \tag{3.16}$$

If we want to compute the Bayes estimator of $\theta = g(\alpha, \lambda)$, under the square loss function L_1, then the corresponding Bayes estimator will be

$$\hat{\theta}_{B1} = E_{posterior}(\theta|\underset{\sim}{X}) = \int_0^\infty \int_0^\infty \theta\, \pi(\alpha, \lambda|\underset{\sim}{X})\, d\alpha\, d\lambda.$$

For the absolute error loss function L_2, the Bayes estimator $\hat{\theta}_{B2}$ will be the median of the posterior distribution θ, *i.e.*

$$\hat{\theta}_{B2} = Med_{posterior}(\theta|\underset{\sim}{X}).$$

If we take the LINEX loss function L_3, then for any $a^* \neq 0$, the Bayes estimator $\hat{\theta}_{B3}$ of θ will be

$$\hat{\theta}_{B3} = \left[E_{posterior}(\theta^{-a^*}|\underset{\sim}{X})\right]^{-\frac{1}{a^*}} = \left[\int_0^\infty \int_0^\infty \theta^{-a^*}\, \pi(\alpha, \lambda|\underset{\sim}{X})\, d\alpha\, d\lambda\right]^{-\frac{1}{a^*}}.$$

It is clear that even if we have a specific form of $\pi_2(\alpha)$, the Bayes estimators $\hat{\theta}_{B1}$, $\hat{\theta}_{B2}$ and $\hat{\theta}_{B3}$, under different loss functions may not be obtained in explicit forms.
Here we develop an algorithm by using the Gibbs sampler method to compute Bayes estimators above and also to construct credible intervals. The Gibbs sampler method, as mentioned in section [4] of chapter 1, needs the conditional distributions $\pi_1(\lambda|\alpha, \underset{\sim}{X})$ and $\pi_2(\alpha|\underset{\sim}{X})$. Note that $\pi_1(\lambda|\alpha, \underset{\sim}{X})$ is obtained in Theorem 1, and for $\pi_2(\alpha|\underset{\sim}{X})$, we state the following theorem :

Theorem 2

The conditional PDF of α given $\underset{\sim}{X}$ is given by

$$\pi_2(\alpha|\underset{\sim}{X}) \propto \pi_2(\alpha)\,\alpha^m \prod_{i=1}^{m} x_{i:m:n}^{\alpha-1} \times \frac{1}{\left(b+\sum\limits_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha}\right)^{a+m}}, \tag{3.17}$$

and it is log-concave.

Proof

From the posterior distribution of α and λ, Eq.(3.16), we have

$$\pi(\lambda,\alpha|\underset{\sim}{X}) \propto L(\alpha,\lambda|\underset{\sim}{X})\;\pi_1(\lambda|\alpha,a,b)\pi_2(\alpha)$$

$$\propto \alpha^m\lambda^m e^{-\lambda\sum\limits_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha}} \prod_{i=1}^{m} x_{i:m:n}^{\alpha-1}\;.\;\lambda^{a-1}e^{-\lambda b}\,\pi_2(\alpha)$$

$$= \pi_2(\alpha)\,\alpha^m \prod_{i=1}^{m} x_{i:m:n}^{\alpha-1}\;.\;\lambda^{a+m-1}\,e^{-\lambda(b+\sum\limits_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha})}.$$

The PDF of α given data is

$$\pi_2(\alpha|\underset{\sim}{X}) = \int_0^{\infty} \pi(\lambda,\alpha|\underset{\sim}{X})\,d\lambda$$

$$\propto \pi_2(\alpha)\alpha^m \prod_{i=1}^{m} x_{i:m:n}^{\alpha-1} \int_0^{\infty} \lambda^{a+m-1} e^{-\lambda(b+\sum\limits_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha})}\,d\lambda$$

$$= \pi_2(\alpha)\alpha^m \prod_{i=1}^{m} x_{i:m:n}^{\alpha-1} \times \frac{\Gamma(a+m)}{(b+\sum\limits_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha})^{a+m}}$$

$$\propto \pi_2(\alpha)\alpha^m \prod_{i=1}^{m} x_{i:m:n}^{\alpha-1} \times \frac{1}{\left(b+\sum\limits_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha}\right)^{a+m}}.$$

Now, consider

$$\ln \pi_2(\alpha|\underset{\sim}{X}) = C + \ln \pi_2(\alpha) + m \ln \alpha + (\alpha - 1) \sum_{i=1}^{m} \ln x_{i:m:n} - (a+m) \ln \left[b + \sum_{i=1}^{m} (1+r_i) x_{i:m:n}^{\alpha} \right],$$

where C is constant.

Suppose that

$$g(\alpha) = b + \sum_{i=1}^{m} (1+r_i) x_{i:m:n}^{\alpha},$$

then

$$g'(\alpha) = \sum_{i=1}^{m} (1+r_i) x_{i:m:n}^{\alpha} \ln x_{i:m:n},$$

and

$$g''(\alpha) = \sum_{i=1}^{m} (1+r_i) x_{i:m:n}^{\alpha} (\ln x_{i:m:n})^2,$$

where $g'(.)$ and $g''(.)$ denote the first and second derivatives of $g(.)$, respectively. It follows that

$$\ln \pi_2(\alpha|\underset{\sim}{X}) = C + \ln \pi_2(\alpha) + m \ln \alpha + (\alpha - 1) \sum_{i=1}^{m} \ln x_{i:m:n} - (a+m) \ln g(\alpha).$$

Now

$$(\ln \pi_2(\alpha|\underset{\sim}{X}))' = (\ln \pi_2(\alpha))' + \frac{m}{\alpha} + \sum_{i=1}^{m} \ln x_{i:m:n} - (a+m) \,.\, \frac{g'(\alpha)}{g(\alpha)}$$

$$(\ln \pi_2(\alpha|\underset{\sim}{X}))'' = (\ln \pi_2(\alpha))'' - \frac{m}{\alpha^2} - (a+m) \,.\, \frac{g(\alpha)g''(\alpha) - (g'(\alpha))^2}{(g(\alpha))^2}$$

Since, $\pi_2(\alpha)$ is assumed log-concave, as mentioned in subsection [1.3.1] of chapter 1, we have $(\ln \pi_2(\alpha))'' < 0$. Observe that $g''(\alpha) > 0$ and so

$$g(\alpha)g''(\alpha) - (g'(\alpha))^2 = bg''(\alpha) + \left[\sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha}\right]\left[\sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha}(\ln x_{i:m:n})^2\right]$$

$$- \left[\sum_{i=1}^{m}(1+r_i)x_{i:m:n}^{\alpha}\ln x_{i:m:n}\right]^2$$

$$= bg''(\alpha) + \sum\sum_{1\leq i<j\leq m}(1+r_i)(1+r_j)x_{i:m:n}^{\alpha}\, x_{j:m:n}^{\alpha}(\ln x_{i:m:n} - \ln x_{j:m:n})^2$$

$$\geq 0 \text{ for } b \geq 0.$$

Therefore $(\ln \pi_2(\alpha|\underset{\sim}{X}))'' < 0$, and thus $\pi_2(\alpha|\underset{\sim}{X})$ is log-concave. One can see this theorem and it's proof in Kundu (2008).

Now by using Theorems 1 and 2, it is possible to generate MCMC samples from the posterior distribution of α and λ, Eq.(3.16), and then use these samples to obtain the Bayes estimators of any function of α and λ, $\theta = g(\alpha, \lambda)$. This enables us to construct the corresponding credible intervals. It may be mentioned that the Bayes estimator of θ depends on the loss function used, while the corresponding credible interval does not depend on the loss function. It just depends on the posterior distribution function.
For this purpose we provide the following algorithm :

Algorithm 1

- **Step 1**
  Generate α from the log-concave density function $\pi_2(\alpha|\underset{\sim}{X})$, Eq.(3.17), using the method proposed by Devroye(1984)[see Apendix[1]].

- **Step 2**
  For each α, generate λ from the posterior density function of λ given α and data, $\pi_1(\lambda|\alpha, \underset{\sim}{X})$, Eq.(3.7).

- **Step 3**
  Repeat steps 1 and 2 M times and obtain MCMC samples $\{(\alpha_i, \lambda_i);\ i = 1, 2, ..., M\}$.

- **Step 4**

  Obtain the Bayes estimate of $\theta = g(\alpha, \lambda)$ with respect to the square error loss function L_1 as

$$\hat{\theta}_{B_1} = \frac{1}{M}\sum_{i=1}^{M} g(\alpha_i, \lambda_i).$$

  Obtain the posterior variance of $\theta = g(\alpha, \lambda)$ as

$$\hat{V}ar(\theta|data) = \frac{1}{M}\sum_{i=1}^{M}(\theta_i - \hat{\theta}_{B_1})^2.$$

  In particular, if $\theta = g(\alpha, \lambda) = \alpha\,(or\,\lambda)$, then

$$\begin{aligned} \hat{\alpha}_{B_1} &= \frac{1}{M}\sum_{i=1}^{M}\alpha_i, \\ \hat{V}ar(\alpha|data) &= \frac{1}{M}\sum_{i=1}^{M}(\alpha_i - \hat{\alpha}_{B_1})^2. \end{aligned}$$

  or

$$\begin{aligned} \hat{\lambda}_{B_1} &= \frac{1}{M}\sum_{i=1}^{M}\lambda_i, \\ \hat{V}ar(\lambda|data) &= \frac{1}{M}\sum_{i=1}^{M}(\lambda_i - \hat{\lambda}_{B_1})^2. \end{aligned}$$

- **Step 5**

  To obtain the Bayes estimate of $\theta = g(\alpha, \lambda)$, under the absolute error loss function L_2, we order $\theta_1, \theta_2, ..., \theta_M$ as $\theta_{(1)} < \theta_{(2)} < ... < \theta_{(M)}$, then the Bayes estimate of $\theta = g(\alpha, \lambda)$ will be

$$\hat{\theta}_{B_2} = Median\left[\theta_{(1)}, \theta_{(2)}, ..., \theta_{(M)}\right].$$

  Obtain the posterior variance of $\theta = g(\alpha, \lambda)$ as

$$\hat{V}ar(\theta|data) = \frac{1}{M}\sum_{i=1}^{M}(\theta_i - \hat{\theta}_{B_2})^2.$$

In particular, if $\theta = g(\alpha, \lambda) = \alpha\,(or\,\lambda)$, we order $\alpha_1, \alpha_2, ..., \alpha_M$ as $\alpha_{(1)} < \alpha_{(2)} < ... < \alpha_{(M)}$ and $\lambda_1, \lambda_2, ..., \lambda_M$ as $\lambda_{(1)} < \lambda_{(2)} < ... < \lambda_{(M)}$, then the Bayes estimate of $\alpha\,(or\,\lambda)$ will be

$$\hat{\alpha}_{B_2} = Median\left[\alpha_{(1)}, \alpha_{(2)}, ..., \alpha_{(M)}\right],$$
$$\hat{V}ar(\alpha|data) = \frac{1}{M}\sum_{i=1}^{M}(\alpha_i - \hat{\alpha}_{B_2})^2.$$

or

$$\hat{\lambda}_{B_2} = Median\left[\lambda_{(1)}, \lambda_{(2)}, ..., \lambda_{(M)}\right],$$
$$\hat{V}ar(\lambda|data) = \frac{1}{M}\sum_{i=1}^{M}(\lambda_i - \hat{\lambda}_{B_2})^2.$$

- **Step 6**
  Obtain the Bayes estimate of $\theta = g(\alpha, \lambda)$ with respect to the LINEX loss function L_3 with $a^* \neq 0$ as

$$\hat{\theta}_{B_3} = \left[\frac{1}{M}\sum_{i=1}^{M}\frac{1}{g^{a*}(\alpha_i, \lambda_i)}\right]^{-\frac{1}{a^*}}.$$

Obtain the posterior variance of $\theta = g(\alpha, \lambda)$ as

$$\hat{V}ar(\theta|data) = \frac{1}{M}\sum_{i=1}^{M}(\theta_i - \hat{\theta}_{B_3})^2.$$

In particular, if $\theta = g(\alpha, \lambda) = \alpha\,(or\,\lambda)$, then

$$\hat{\alpha}_{B_3} = \left[\frac{1}{M}\sum_{i=1}^{M}\frac{1}{\alpha_i^{a*}}\right]^{-\frac{1}{a^*}},$$
$$\hat{V}ar(\alpha|data) = \frac{1}{M}\sum_{i=1}^{M}(\alpha_i - \hat{\alpha}_{B_3})^2.$$

or

$$\hat{\lambda}_{B_3} = \left[\frac{1}{M}\sum_{i=1}^{M}\frac{1}{\lambda_i^{a*}}\right]^{-\frac{1}{a*}},$$

$$\hat{V}ar(\lambda|data) = \frac{1}{M}\sum_{i=1}^{M}(\lambda_i - \hat{\lambda}_{B_3})^2.$$

- **Step 7**
  To compute the credible interval of $\theta = g(\alpha, \lambda)$, we order $\theta_1, \theta_2, ..., \theta_M$ as $\theta_{(1)} < \theta_{(2)} < ... < \theta_{(M)}$. Then the $(1-\beta)100\%$ symmetric credible interval of θ is given by

$$\left(\theta_{([\frac{M\beta}{2}])}\,,\,\theta_{([M(1-\frac{\beta}{2})])}\right),$$

  where $[x]$ denotes the largest integer less than or equal x.
  In particular, if $\theta = g(\alpha, \lambda) = \alpha$, we order $\alpha_1, \alpha_2, ..., \alpha_M$ as $\alpha_{(1)} < \alpha_{(2)} < ... < \alpha_{(M)}$ and then the $(1-\beta)100\%$ symmetric credible interval of α is given by

$$\left(\alpha_{([\frac{M\beta}{2}])}\,,\,\alpha_{([M(1-\frac{\beta}{2})])}\right),$$

  and in the same way, the $(1-\beta)100\%$ symmetric credible interval of λ is given by

$$\left(\lambda_{([\frac{M\beta}{2}])}\,,\,\lambda_{([M(1-\frac{\beta}{2})])}\right).$$

3.3 Bayes Prediction

In this section, under different loss functions described in subsection [1.3.1] of chapter 1 and based on observed progressive type II censored data, we will derive the posterior predictive density, which is necessary to obtain the Bayes predictors, of missing and future progressive type II censored data when one-sample and two-sample prediction problems are used. As well as we will find the predictive survival function, which is necessary to obtain bounds, of missing and future progressive type II censored data, when both one-sample and two-sample prediction problems are used.

3.3.1 One-Sample Prediction

Let $\underset{\sim}{x} = (x_{1:m:n}\,, x_{2:m:n}\,, \ldots\,, x_{m:m:n})$ be the observed progressive type II censored sample of size m from a sample of size n, drawn from a Weibull distribution, with progressive censoring scheme $(\,r_1\,,\,,r_2\,,\,\ldots\,,\,r_m)$, where m is the number of censoring stages. Based on the observed progressive type II censored sample $\underset{\sim}{x}$, our aim is to obtain the Bayes predictive estimator, under different loss functions: L_1, L_2 and L_3 with $a^* \neq 0$, as well as constructing the predictive interval of the jth order statistic, $Y = Y_{k:r_j}$ $(k = 1, 2, \ldots, r_j\,;\ j = 1, 2, \ldots, m)$, from a sample of size r_j removed items at stage j. This is the one-sample prediction technique.

To obtain the Bayes predictive estimator, we need to define the posterior predictive density of $Y = Y_{k:r_j}$ given the observed progressive censored sample $\underset{\sim}{x}$. Based on Eq.(1.5), the posterior predictive density of $Y = Y_{k:r_j}$ can be written as

$$\pi_Y(y|\underset{\sim}{x}) = \int_0^\infty \int_0^\infty f_{Y|\underset{\sim}{x}}(y|\alpha,\lambda)\,\pi(\alpha,\lambda|\underset{\sim}{x})\,d\alpha\,d\lambda,\ y > x_{j:m:n},$$

where $f_{Y|\underset{\sim}{x}}(y|\alpha,\lambda)$ is the conditional density function of $Y = Y_{k:r_j}$ given α, λ and the data $\underset{\sim}{x}$.
Using the Markovian property of progressively type II censored order statistics, see Balakrishnan and Aggarwala (2000), the conditional PDF of $Y = Y_{k:r_j}$ given $\underset{\sim}{x}$ is just the conditional PDF of $Y = Y_{k:r_j}$ given $x_{j:m:n}$, *i.e.*

$$f_{Y|\underset{\sim}{x}}(y|\alpha,\lambda) = f_{Y|x_{j:m:n}}(y|\alpha,\lambda).$$

Using the Markovian property and the fact that the conditional density of $Y = Y_{k:r_j}$ is the same as the density of the kth order statistic from a sample of size r_j with density $\frac{f(y)}{1-F(x_{j:m:n})}$, $y > x_{j:m:n}$, then the conditional density of $Y = Y_{k:r_j}$ given α, λ and the data $\underset{\sim}{x}$ becomes

$$\begin{aligned} f_{Y|\underset{\sim}{x}}(y|\alpha,\lambda) &= f_{Y|x_{j:m:n}}(y|\alpha,\lambda) \\ &= C\left[F(y|\alpha,\lambda) - F(x_{j:m:n}|\alpha,\lambda)\right]^{k-1}\left[1-F(y|\alpha,\lambda)\right]^{r_j-k} \\ &\quad \times \frac{f(y|\alpha,\lambda)}{\left[1-F(x_{j:m:n}|\alpha,\lambda)\right]^{r_j}}, y > x_{j:m:n}, \end{aligned}$$

where $C = \frac{r_j!}{(k-1)!\,(r_j-k)!}$.

Based on the PDF (1.1) and CDF (1.2), we have

$$f_{Y|\underset{\sim}{x}}(y|\alpha,\lambda) = C\left[e^{-\lambda x^{\alpha}_{j:m:n}} - e^{-\lambda y^{\alpha}}\right]^{k-1} e^{-\lambda(r_j-k)y^{\alpha}} \times \frac{\alpha\lambda y^{\alpha-1}\, e^{-\lambda y^{\alpha}}}{e^{-\lambda r_j x^{\alpha}_{j:m:n}}}.$$

By using the binomial expansion, we have

$$\begin{aligned} f_{Y|\underset{\sim}{x}}(y|\alpha,\lambda) &= C\left[\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^{k-i-1}e^{-\lambda i x^{\alpha}_{j:m:n}}e^{-\lambda(k-i-1)y^{\alpha}}\right]e^{-\lambda(r_j-k)y^{\alpha}} \\ &\quad \times \alpha\lambda y^{\alpha-1}\, e^{-\lambda y^{\alpha}}\, e^{\lambda r_j x^{\alpha}_{j:m:n}} \\ &= C\alpha\lambda\sum_{i=0}^{k-1}\left[\binom{k-1}{i}(-1)^{k-i-1}e^{-\lambda(i-r_j)x^{\alpha}_{j:m:n}} \times y^{\alpha-1}\, e^{-\lambda(r_j-i)y^{\alpha}}\right]. \end{aligned} \tag{3.18}$$

The posterior predictive density of $Y = Y_{k:r_j}$ at any point $y > x_{j:m:n}$ is then

$$\begin{aligned} f^{P}_{Y|\underset{\sim}{x}}(y|\alpha,\lambda) &= E_{posterior}\left[f_{Y|x_{j:m:n}}(y|\alpha,\lambda)\right] \\ &= C\int_0^{\infty}\int_0^{\infty}\left[\alpha\lambda\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^{k-i-1}e^{-\lambda(i-r_j)x^{\alpha}_{j:m:n}} \times y^{\alpha-1}\, e^{-\lambda(r_j-i)y^{\alpha}}\right] \\ &\quad \times\ \pi(\alpha,\lambda|\underset{\sim}{x})\, d\alpha d\lambda. \end{aligned} \tag{3.19}$$

Under the square error loss function L_1, the Bayes predictive estimator of $Y = Y_{k:r_j}$ can be obtained as

$$\begin{aligned}
Y_{k:r_j}^{BP1} &= E_{f^P}(Y|\underset{\sim}{x}) \\
&= \int_{x_{j:m:n}}^{\infty} y f_{Y|\underset{\sim}{x}}^{P}(y|\alpha,\lambda)\, dy \\
&= C\int_{x_{j:m:n}}^{\infty} \int_0^{\infty}\int_0^{\infty} \left[\alpha\lambda \sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^{k-i-1} e^{-\lambda(i-r_j)x_{j:m:n}^{\alpha}} \times y^{\alpha}\, e^{-\lambda(r_j-i)y^{\alpha}}\right] \\
&\quad \times\ \pi(\alpha,\lambda|\underset{\sim}{x})\, d\alpha d\lambda\ dy.
\end{aligned}$$

Based on MCMC samples $\{(\alpha_l,\lambda_l);\ l = 1,2,...,M\}$ obtained by using the Gibbs sampling method, the simulation estimate $\hat{Y}_{k:r_j}^{BP1}$ of $Y = Y_{k:r_j}$ will be

$$\hat{Y}_{k:r_j}^{BP1} = C\int_{x_{j:m:n}}^{\infty} \left[\frac{1}{M}\sum_{l=1}^{M}\alpha_l\lambda_l \sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^{k-i-1} e^{-\lambda_l(i-r_j)x_{j:m:n}^{\alpha_l}} \times y^{\alpha_l}\, e^{-\lambda_l(r_j-i)y^{\alpha_l}}\right] dy$$

$$= \frac{C}{M}\sum_{l=1}^{M}\alpha_l\lambda_l \sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^{k-i-1} e^{-\lambda_l(i-r_j)x_{j:m:n}^{\alpha_l}} \times \int_{x_{j:m:n}}^{\infty} y^{\alpha_l}\, e^{-\lambda_l(r_j-i)y^{\alpha_l}}\, dy$$

By making the transformation $u = \lambda_l(r_j - i)y^{\alpha_l}$ and converting the integrand to the PDF of gamma distribution, and by using Eq.(3.13), the simulation estimate $\hat{Y}_{k:r_j}^{BP1}$ of $Y = Y_{k:r_j}$ becomes

$$\begin{aligned}
\hat{Y}_{k:r_j}^{BP1} &= \frac{C}{M}\sum_{l=1}^{M}\alpha_l\lambda_l \sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^{k-i-1} e^{-\lambda_l(i-r_j)x_{j:m:n}^{\alpha_l}} \times \frac{\Gamma\left(\frac{1}{\alpha_l}+1,\ \lambda_l(r_j-i)x_{j:m:n}^{\alpha_l}\right)}{\alpha_l(\lambda_l(r_j-i))^{\frac{1}{\alpha_l}+1}} \\
&= \frac{C}{M}\sum_{l=1}^{M}\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^{k-i-1} e^{-\lambda_l(i-r_j)x_{j:m:n}^{\alpha_l}} \times \frac{\Gamma\left(\frac{1}{\alpha_l}+1,\ \lambda_l(r_j-i)x_{j:m:n}^{\alpha_l}\right)}{\lambda_l^{\frac{1}{\alpha_l}}(r_j-i)^{\frac{1}{\alpha_l}+1}}.
\end{aligned} \tag{3.20}$$

In particular, the simulation estimate of the first unobserved value in any censoring stage j, $Y_{1:r_j}$, can be obtained by setting $k = 1$ in Eq.(3.20) as follows

$$\hat{Y}_{1:r_j}^{BP1} = \frac{C}{M}\sum_{l=1}^{M} e^{\lambda_l r_j x_{j:m:n}^{\alpha_l}} \times \frac{\Gamma\left(\frac{1}{\alpha_l}+1,\ \lambda_l\, r_j\, x_{j:m:n}^{\alpha_l}\right)}{\lambda_l^{\frac{1}{\alpha_l}}\, r_j^{\frac{1}{\alpha_l}+1}}.$$

Under the absolute error loss function L_2, the corresponding Bayes predictive estimator of $Y = Y_{k:r_j}$ $(k = 1, 2, \ldots, r_j\,;\ j = 1, 2, \ldots, m)$, denoted by $Y_{k:r_j}^{BP2}$ is the median of the posterior predictive density of $Y = Y_{k:r_j}$, Eq.(3.19), which is obtained by solving the following equation with respect to $Y_{k:r_j}^{BP2}$

$$\int_{x_{j:m:n}}^{Y_{k:r_j}^{BP2}} f_{Y|\underset{\sim}{x}}^{P}(y|\alpha, \lambda)\, dy = \frac{1}{2}. \tag{3.21}$$

Equation (3.21) is equivalent to

$$\int_{Y_{k:r_j}^{BP2}}^{\infty} f_{Y|\underset{\sim}{x}}^{P}(y|\alpha, \lambda)\, dy = \frac{1}{2}. \tag{3.22}$$

Based on the posterior predictive density of $Y = Y_{k:r_j}$, Eq.(3.19), Eq.(3.22) is equivalent to

$$C \int_{Y_{k:r_j}^{BP2}}^{\infty} \left(\int_0^{\infty}\int_0^{\infty} \left[\alpha\lambda \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^{k-i-1} e^{-\lambda(i-r_j)x_{j:m:n}^{\alpha}} \times y^{\alpha-1}\, e^{-\lambda(r_j-i)y^{\alpha}} \right] \pi(\alpha, \lambda|\underset{\sim}{x})\, d\alpha d\lambda \right) dy = \frac{1}{2}.$$

As before, based on MCMC samples $\{(\alpha_l, \lambda_l);\ l = 1, 2, ..., M\}$, the simulation estimate $\hat{Y}_{k:r_j}^{BP2}$ of $Y = Y_{k:r_j}$ can be obtained by solving, with respect to $\hat{Y}_{k:r_j}^{BP2}$, the following equation

$$C \int_{\hat{Y}_{k:r_j}^{BP2}}^{\infty} \left(\frac{1}{M}\sum_{l=1}^{M} \alpha_l\lambda_l \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^{k-i-1} e^{-\lambda_l(i-r_j)x_{j:m:n}^{\alpha_l}} \times y^{\alpha_l-1}\, e^{-\lambda_l(r_j-i)y^{\alpha_l}} \right) dy = \frac{1}{2},$$

or

$$\frac{C}{M}\sum_{l=1}^{M}\alpha_l\lambda_l\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^{k-i-1}e^{-\lambda_l(i-r_j)x_{j:m:n}^{\alpha_l}}\times\int\limits_{\hat{Y}_{k:r_j}^{BP2}}^{\infty}y^{\alpha_l-1}\,e^{-\lambda_l(r_j-i)y^{\alpha_l}}dy=\frac{1}{2}.$$

By making the transformation $u=\lambda_l(r_j-i)y^{\alpha_l}$, we have

$$\frac{C}{M}\sum_{l=1}^{M}\alpha_l\lambda_l\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^{k-i-1}e^{-\lambda_l(i-r_j)x_{j:m:n}^{\alpha_l}}\times\left[\frac{1}{\alpha_l\lambda_l(r_j-i)}\int\limits_{\lambda_l(r_j-i)(\hat{Y}_{k:r_j}^{BP2})^{\alpha_l}}^{\infty}e^{-u}\,du\right]=\frac{1}{2},$$

or

$$\frac{C}{M}\sum_{l=1}^{M}\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^{k-i-1}e^{-\lambda_l(i-r_j)x_{j:m:n}^{\alpha_l}}\times\frac{e^{-\lambda_l(r_j-i)(\hat{Y}_{k:r_j}^{BP2})^{\alpha_l}}}{r_j-i}=\frac{1}{2}. \qquad (3.23)$$

In particular, the simulation estimate of $Y_{1:r_j}$ can be obtained by setting $k=1$ in Eq.(3.23) and solving, with respect to $\hat{Y}_{1:r_j}^{BP2}$, the following equation

$$\frac{C}{M}\sum_{l=1}^{M}e^{\lambda_l r_j x_{j:m:n}^{\alpha_l}}\times\frac{e^{-\lambda_l r_j(\hat{Y}_{1:r_j}^{BP2})^{\alpha_l}}}{r_j}=\frac{1}{2}.$$

Under the LINEX loss function L_3 with $a^*\neq 0$, the corresponding Bayes predictive estimator $Y_{k:r_j}^{BP3}$ of $Y_{k:r_j}$ can be obtained as follows

$$\begin{aligned}
Y_{k:r_j}^{BP3} &= \left[E_{f^P}(Y^{-a^*}|\underset{\sim}{x})\right]^{-\frac{1}{a^*}}\\
&= \left[\int\limits_{x_{j:m:n}}^{\infty}y^{-a^*}f_{Y|\underset{\sim}{x}}^{P}(y|\alpha,\lambda)\,dy\right]^{-\frac{1}{a^*}}\\
&= \Bigg[C\int_{x_{j:m:n}}^{\infty}\int\limits_{0}^{\infty}\int\limits_{0}^{\infty}\alpha\lambda\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^{k-i-1}e^{-\lambda(i-r_j)x_{j:m:n}^{\alpha}}\times y^{\alpha-a^*-1}\,e^{-\lambda(r_j-i)y^{\alpha}}\\
&\qquad\times\,\pi(\alpha,\lambda|\underset{\sim}{x})\,d\alpha d\lambda\;dy\Bigg]^{-\frac{1}{a^*}}.
\end{aligned}$$

As before, based on MCMC samples $\{(\alpha_l, \lambda_l);\ l = 1, 2, ..., M\}$, the simulation estimate $\hat{Y}^{BP3}_{k:r_j}$ of $Y = Y_{k:r_j}$ will be

$$\hat{Y}^{BP3}_{k:r_j} = \left[C \int_{x_{j:m:n}}^{\infty} \frac{1}{M} \sum_{l=1}^{M} \alpha_l \lambda_l \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^{k-i-1} e^{-\lambda_l (i-r_j) x_{j:m:n}^{\alpha_l}} \right.$$
$$\left. \times\ y^{\alpha_l - a^* - 1}\, e^{-\lambda_l (r_j - i) y^{\alpha_l}}\, dy \right]^{-\frac{1}{a^*}},$$
$$= \left[\frac{C}{M} \sum_{l=1}^{M} \alpha_l \lambda_l \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^{k-i-1} e^{-\lambda_l (i-r_j) x_{j:m:n}^{\alpha_l}} \right.$$
$$\left. \times \int_{x_{j:m:n}}^{\infty} y^{\alpha_l - a^* - 1}\, e^{-\lambda_l (r_j - i) y^{\alpha_l}}\, dy \right]^{-\frac{1}{a^*}}.$$

By making the transformation $u = \lambda_l (r_j - i) y^{\alpha_l}$ and converting the integrand to the PDF of gamma distribution, and by using Eq.(3.13), the simulation estimate $\hat{Y}^{BP3}_{k:r_j}$ of $Y = Y_{k:r_j}$ becomes

$$\hat{Y}^{BP3}_{k:r_j} = \left[\frac{C}{M} \sum_{l=1}^{M} \alpha_l \lambda_l \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^{k-i-1} e^{-\lambda_l (i-r_j) x_{j:m:n}^{\alpha_l}} \times \frac{\Gamma\left(1 - \frac{a^*}{\alpha_l},\ \lambda_l (r_j - i) x_{j:m:n}^{\alpha_l}\right)}{\alpha_l (\lambda_l (r_j - i))^{1 - \frac{a^*}{\alpha_l}}} \right]^{-\frac{1}{a^*}}$$
$$= \left[\frac{C}{M} \sum_{l=1}^{M} \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^{k-i-1} e^{-\lambda_l (i-r_j) x_{j:m:n}^{\alpha_l}} \times \frac{\Gamma\left(1 - \frac{a^*}{\alpha_l},\ \lambda_l (r_j - i) x_{j:m:n}^{\alpha_l}\right)}{\lambda_l^{-\frac{a^*}{\alpha_l}} (r_j - i)^{1 - \frac{a^*}{\alpha_l}}} \right]^{-\frac{1}{a^*}}. \tag{3.24}$$

In particular, the simulation estimate $\hat{Y}^{BP3}_{1:r_j}$ of $Y = Y_{1:r_j}$, can be obtained by setting $k = 1$ in Eq.(3.24) as follows

$$\hat{Y}^{BP3}_{1:r_j} = \left[\frac{C}{M} \sum_{l=1}^{M} e^{\lambda_l r_j x_{j:m:n}^{\alpha_l}} \times \frac{\Gamma\left(1 - \frac{a^*}{\alpha_l},\ \lambda_l r_j x_{j:m:n}^{\alpha_l}\right)}{\lambda_l^{-\frac{a^*}{\alpha_l}} r_j^{1 - \frac{a^*}{\alpha_l}}} \right]^{-\frac{1}{a^*}}.$$

To obtain prediction bounds on $Y = Y_{k:r_j}$ $(k = 1, 2, \ldots, r_j\,;\ j = 1, 2, \ldots, m)$, we need to find the predictive survival function of $Y = Y_{k:r_j}$ at any point $y > x_{j:m:n}$. Based on Eq.(3.18), the survival function of $Y = Y_{k:r_j}$ is defined by

$$\begin{aligned}
S_{Y|\underset{\sim}{x}}(y|\alpha,\lambda) &= \int_y^{\infty} f_{Y|\underset{\sim}{x}}(z|\alpha,\lambda)\,dz \\
&= \int_y^{\infty} C\alpha\lambda \sum_{i=0}^{k-1}\left[\binom{k-1}{i}(-1)^{k-i-1}e^{-\lambda(i-r_j)x^{\alpha}_{j:m:n}}\, z^{\alpha-1}\, e^{-\lambda(r_j-i)z^{\alpha}}\right]dz \\
&= C\alpha\lambda \sum_{i=0}^{k-1}\left[\binom{k-1}{i}(-1)^{k-i-1}e^{-\lambda(i-r_j)x^{\alpha}_{j:m:n}} \int_y^{\infty} z^{\alpha-1}\, e^{-\lambda(r_j-i)z^{\alpha}}dz\right]
\end{aligned}$$

By making the transformation $u = \lambda_l(r_j - i)z^{\alpha_l}$, the survival function becomes

$$\begin{aligned}
S_{Y|\underset{\sim}{x}}(y|\alpha,\lambda) &= C\alpha\lambda \sum_{i=0}^{k-1}\left[\binom{k-1}{i}(-1)^{k-i-1}e^{-\lambda(i-r_j)x^{\alpha}_{j:m:n}}\, \frac{1}{\alpha\lambda(r_j-i)} \int_{\lambda(r_j-i)y^{\alpha}}^{\infty} e^{-u}du\right] \\
&= C \sum_{i=0}^{k-1}\left[\binom{k-1}{i}(-1)^{k-i-1}e^{-\lambda(i-r_j)x^{\alpha}_{j:m:n}}\, \frac{e^{-\lambda(r_j-i)y^{\alpha}}}{r_j-i}\right]. \qquad (3.25)
\end{aligned}$$

Under different loss functions: L_1, L_2 and L_3 with $a^* \neq 0$, the predictive survival function of $Y = Y_{k:r_j}$ can be obtained, respectively, as follows :

$$\begin{aligned}
S^{P}_{Y|\underset{\sim}{x}}(y|\alpha,\lambda) &= E_{posterior}\left(S_{Y|\underset{\sim}{x}}(y|\alpha,\lambda)\right) \\
&= \int_0^{\infty}\int_0^{\infty} C \sum_{i=0}^{k-1}\left[\binom{k-1}{i}(-1)^{k-i-1}e^{-\lambda(i-r_j)x^{\alpha}_{j:m:n}}\, \frac{e^{-\lambda(r_j-i)y^{\alpha}}}{r_j-i}\right]\pi(\alpha,\lambda|\underset{\sim}{x})\,d\alpha\,d\lambda
\end{aligned}$$

$$S^{P}_{Y|\underset{\sim}{x}}(y|\alpha,\lambda) = Med_{posterior}\left[S_{Y|\underset{\sim}{x}}(y|\alpha,\lambda)\right]$$

$$\begin{aligned}
S^{P}_{Y|\underset{\sim}{x}}(y|\alpha,\lambda) &= \left[E_{posterior}\left(S_{Y|\underset{\sim}{x}}(y|\alpha,\lambda)\right)^{-a^*}\right]^{-\frac{1}{a^*}} \\
&= \left[\int_0^{\infty}\int_0^{\infty}\left(C \sum_{i=0}^{k-1}\left[\binom{k-1}{i}(-1)^{k-i-1}e^{-\lambda(i-r_j)x^{\alpha}_{j:m:n}}\, \frac{e^{-\lambda(r_j-i)y^{\alpha}}}{r_j-i}\right]\right)^{-a^*}\pi(\alpha,\lambda|\underset{\sim}{x})\,d\alpha\,d\lambda\right]^{-\frac{1}{a^*}}. \qquad (3.26)
\end{aligned}$$

It is clear that the equations in (3.26) can't be expressed in a closed form and hence it can't be evaluated analytically. By using the MCMC samples $\{(\alpha_l, \lambda_l);\ l = 1, 2, ..., M\}$ obtained by the Gibbs sampling procedure, the simulation estimator for the predictive survival function of $Y = Y_{k:r_j}$ under L_1, is given by

$$\hat{S}^P_{Y|\underset{\sim}{x}}(y) = \frac{C}{M} \sum_{l=1}^{M} \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^{k-i-1} e^{-\lambda_l (i-r_j) x^{\alpha_l}_{j:m:n}} \times \frac{e^{-\lambda_l (r_j - i) y^{\alpha_l}}}{r_j - i}.$$

Under the absolute error loss function L_2, the simulation estimator for the predictive survival function of $Y = Y_{k:r_j}$, can be obtained by using the following algorithm :

Algorithm 2

- **Step 1**
  Evaluate $S = S_{Y|\underset{\sim}{x}}(y|\alpha, \lambda)$, Eq.(3.25), at each sample (α_l, λ_l) for $l = 1, 2, ..., M$, to get $S_1\ ,\ S_2\ ,\ ...\ ,\ S_M$

- **Step 2**
  Order $S_1\ ,\ S_2\ ,\ ...\ ,\ S_M$ as $S_{(1)}\ <\ S_{(2)}\ <\ ...\ <\ S_{(M)}$

- **Step 3**
  The simulation estimator for the predictive survival function of $Y = Y_{k:r_j}$ is given by

$$\hat{S}^P_{Y|\underset{\sim}{x}}(y) = Median\left[S_{(1)}\ ,\ S_{(2)}\ ,\ ...\ ,\ S_{(M)}\right].$$

Under the LINEX loss function L_3 with $a^* \neq 0$, the simulation estimator for the predictive survival function of $Y = Y_{k:r_j}$, can be obtained as

$$\hat{S}^P_{Y|\underset{\sim}{x}}(y) = \left[\frac{C}{M} \sum_{l=1}^{M} \left(\sum_{i=0}^{k-1} \left[\binom{k-1}{i} (-1)^{k-i-1} e^{-\lambda_l (i-r_j) x^{\alpha_l}_{j:m:n}}\ \frac{e^{-\lambda_l (r_j - i) y^{\alpha_l}}}{r_j - i} \right] \right)^{-a^*} \right]^{-\frac{1}{a^*}}.$$

Another important aspect of prediction is to construct a two-sided interval for $Y = Y_{k:r_j}$ $(k = 1, 2, \ldots, r_j\,;\, j = 1, 2, \ldots, m)$.
A $(1-\beta)\%$ predictive interval (PI) of $Y = Y_{k:r_j}$, under different loss functions $L_1,\ L_2$ and L_3, can be found by solving the non-linear equations (3.27) and (3.28) for the lower bound L and upper bound U :

$$P(Y > L|\underset{\sim}{x}) = 1 - \frac{\beta}{2} \;\Leftrightarrow\; \hat{S}^{P}_{Y|\underset{\sim}{x}}(L) = 1 - \frac{\beta}{2}, \tag{3.27}$$

$$P(Y > U|\underset{\sim}{x}) = \frac{\beta}{2} \;\Leftrightarrow\; \hat{S}^{P}_{Y|\underset{\sim}{x}}(U) = \frac{\beta}{2}. \tag{3.28}$$

We need to apply a suitable numerical method to solve these non-linear equations as they can't be solved analytically.

3.3.2 Two-Sample Prediction

Suppose that $\underset{\sim}{X} = (X_{1:m_1:n_1}\,,\, X_{2:m_1:n_1}\,,\, \ldots\,,\, X_{m_1:m_1:n_1})$ is a progressive type II censored sample of size m_1 from a sample of size n_1 drawn from a Weibull distribution, with progressive censoring scheme $(r_1\,,\, r_2\,,\, \ldots\,,\, r_{m_1})$.
Suppose also that $\underset{\sim}{Y} = (Y_{1:m_2:n_2}\,,\, Y_{2:m_2:n_2}\,,\, \ldots\,,\, Y_{m_2:m_2:n_2})$ is a second (unobserved) independent progressive type II censored sample of size m_2 from a sample of size n_2 drawn from the same population, with progressive censoring scheme $(s_1\,,\, s_2\,,\, \ldots\,,\, s_{m_2})$.
The first sample is referred to the "informative" (observed) sample, while the second one is referred to the (future) sample. Based on an informative progressive type II censored sample, our aim is to predict the kth order statistic in the future sample; $Y_{k:m_2:n_2}\,,\, k = 1, 2, \ldots, m_2$, and also to construct the predictive interval for $Y_{k:m_2:n_2}$.
This is the two-sample prediction technique.
To obtain the Bayes predictive estimator of $Y = Y_{k:m_2:n_2}\,,\, k = 1, 2, \ldots, m_2$, under different loss functions used in the previous section, we need the posterior predictive density of $Y = Y_{k:m_2:n_2}$.
The posterior predictive density of $Y = Y_{k:m_2:n_2}$ can be obtained as follows :

The PDF of $Y = Y_{k:m_2:n_2}$, $k = 1, 2, \ldots, m_2$, is given by [see for instance, Balakrishnan et al. (2001) or Kamps and Carmer (2001)]

$$g_{(k)}(y|\alpha,\lambda) = C_{k-1} f(y|\alpha,\lambda) \sum_{i=1}^{k} a_{i,k} \left(1 - F(y|\alpha,\lambda)\right)^{\gamma_i - 1}, \tag{3.29}$$

where

$$\begin{aligned} \gamma_i &= n_2 - \left(\sum_{j=1}^{i-1} s_j\right) - i + 1\,,\ \gamma_1 = n_2, \\ C_{k-1} &= \prod_{i=1}^{k} \gamma_i, \\ a_{i,k} &= \prod_{\substack{j=1 \\ j \neq i}}^{k} \frac{1}{\gamma_j - \gamma_i}\ , \text{ for } 1 \leq i \leq k \leq m_2\ \text{ , and for } \ k = 1,\ a_{1,k} = 1. \end{aligned}$$

Based on the PDF (1.1) and CDF (1.2) of the Weibull distribution, the PDF of $Y = Y_{k:m_2:n_2}$, $k = 1, 2, \ldots, m_2$, Eq.(3.29), becomes

$$g_{(k)}(y|\alpha,\lambda) = C_{k-1} \alpha\lambda\, y^{\alpha-1} \sum_{i=1}^{k} a_{i,k}\, e^{-\lambda\gamma_i\, y^{\alpha}}. \tag{3.30}$$

Based on a progressive type II censored (informative) sample $\underset{\sim}{X}$ and Eq.(1.5), the posterior predictive density function of $Y = Y_{k:m_2:n_2}$, $k = 1, 2, \ldots, m_2$, is given by

$$\begin{aligned} g^{P}_{(k)}(y|\alpha,\lambda) &= E_{posterior}\left[\, g_{(k)}(y|\alpha,\lambda)\right] \\ &= \int_0^{\infty}\!\!\int_0^{\infty} g_{(k)}(y|\alpha,\lambda)\, \pi(\alpha,\lambda|\underset{\sim}{X})\, d\alpha\, d\lambda \\ &= \int_0^{\infty}\!\!\int_0^{\infty} C_{k-1}\alpha\lambda\, y^{\alpha-1} \sum_{i=1}^{k} a_{i,k}\, e^{-\lambda\gamma_i\, y^{\alpha}}\ \pi(\alpha,\lambda|\underset{\sim}{X})\, d\alpha\, d\lambda. \end{aligned} \tag{3.31}$$

Under the square error loss function L_1, the Bayes predictive estimator of $Y = Y_{k:m_2:n_2}$ can be obtained as

$$\begin{aligned} Y^{BP1} &= E_{g^P_{(k)}}(Y|\underset{\sim}{X}) \\ &= \int_0^\infty y\, g^P_{(k)}(y|\alpha,\lambda)\,dy \\ &= \int_0^\infty \left[\int_0^\infty \int_0^\infty C_{k-1}\alpha\lambda\, y^\alpha \sum_{i=1}^{k} a_{i,k}\, e^{-\lambda\gamma_i\, y^\alpha}\, \pi(\alpha,\lambda|\underset{\sim}{X})\, d\alpha\, d\lambda \right] dy. \end{aligned}$$

Based on MCMC samples $\{(\alpha_l,\lambda_l);\ l=1,2,...,M\}$, the simulation estimate $\hat{Y}^{BP1}$of $Y=Y_{k:m_2:n_2}$ will be

$$\begin{aligned} \hat{Y}^{BP1} &= \int_0^\infty \left[\frac{1}{M}\sum_{l=1}^{M} C_{k-1}\alpha_l\lambda_l\, y^{\alpha_l} \sum_{i=1}^{k} a_{i,k}\, e^{-\lambda_l\gamma_i\, y^{\alpha_l}} \right] dy \\ &= \frac{C_{k-1}}{M}\sum_{l=1}^{M}\alpha_l\lambda_l \sum_{i=1}^{k} a_{i,k} \int_0^\infty y^{\alpha_l}\, e^{-\lambda_l\gamma_i\, y^{\alpha_l}}\, dy. \end{aligned}$$

By making the transformation $u=\lambda_l\gamma_i y^{\alpha_l}$, and converting the integrand to the PDF of gamma distribution, the simulation estimate of $Y=Y_{k:m_2:n_2}$ becomes

$$\begin{aligned} \hat{Y}^{BP1} &= \frac{C_{k-1}}{M}\sum_{l=1}^{M}\alpha_l\lambda_l \sum_{i=1}^{k} a_{i,k} \times \frac{\Gamma(1+\frac{1}{\alpha_l})}{\alpha_l(\lambda_l\gamma_i)^{1+\frac{1}{\alpha_l}}} \\ &= \frac{C_{k-1}}{M}\sum_{l=1}^{M}\sum_{i=1}^{k} a_{i,k} \times \frac{\Gamma(1+\frac{1}{\alpha_l})}{(\lambda_l)^{\frac{1}{\alpha_l}}(\gamma_i)^{1+\frac{1}{\alpha_l}}}. \end{aligned} \tag{3.32}$$

Under the absolute error loss function L_2, the corresponding Bayes predictive estimator of $Y=Y_{k:m_2:n_2}$ denoted by Y^{BP2}, is the median of the posterior predictive density, Eq.(3.31), which is obtained by solving the following equation with respect to Y^{BP2}

$$\int_0^{Y^{BP2}} g^P_{(k)}(y|\alpha,\lambda)\,dy = \frac{1}{2},$$

or equivalently to

$$\int_{Y^{BP2}}^\infty g^P_{(k)}(y|\alpha,\lambda)\,dy = \frac{1}{2}. \tag{3.33}$$

Based on Eq.(3.31), Eq.(3.33) is equivalent to

$$\int_{Y^{BP2}}^{\infty}\left[\int_0^{\infty}\int_0^{\infty} C_{k-1}\alpha\lambda\, y^{\alpha-1}\sum_{i=1}^{k} a_{i,k}\, e^{-\lambda\gamma_i\, y^{\alpha}}\,\pi(\alpha,\lambda|\underset{\sim}{X})\, d\alpha\, d\lambda\right] dy = \frac{1}{2}.$$

As before, based on MCMC samples $\{(\alpha_l, \lambda_l);\ l = 1, 2, ..., M\}$, the simulation estimate $\hat{Y}^{BP2}$ of $Y = Y_{k:m_2:n_2}$ can be obtained by solving, with respect to $\hat{Y}^{BP2}$, the following equation

$$\int_{\hat{Y}^{BP2}}^{\infty}\left[\frac{1}{M}\sum_{l=1}^{M} C_{k-1}\alpha_l\lambda_l\, y^{\alpha_l-1}\sum_{i=1}^{k} a_{i,k}\, e^{-\lambda_l\gamma_i\, y^{\alpha_l}}\right] dy = \frac{1}{2},$$

which is equivalent to solving

$$\frac{C_{k-1}}{M}\sum_{l=1}^{M}\alpha_l\lambda_l\sum_{i=1}^{k} a_{i,k}\int_{\hat{Y}^{BP2}}^{\infty} y^{\alpha_l-1}\, e^{-\lambda_l\gamma_i\, y^{\alpha_l}}\, dy = \frac{1}{2}. \tag{3.34}$$

By making the transformation $u = \lambda_l\gamma_i y^{\alpha_l}$, Eq.(3.34) is equivalent to

$$\frac{C_{k-1}}{M}\sum_{l=1}^{M}\alpha_l\lambda_l\sum_{i=1}^{k} a_{i,k}\times\frac{e^{-\lambda_l\gamma_i(\hat{Y}^{BP2})^{\alpha_l}}}{\alpha_l\lambda_l\gamma_i} = \frac{1}{2},$$

or equivalently to

$$\frac{C_{k-1}}{M}\sum_{l=1}^{M}\sum_{i=1}^{k}\frac{a_{i,k}\, e^{-\lambda_l\gamma_i(\hat{Y}^{BP2})^{\alpha_l}}}{\gamma_i} = \frac{1}{2}. \tag{3.35}$$

Under the LINEX loss function L_3 with $a^* \neq 0$, the corresponding Bayes predictive estimator Y^{BP3} of $Y = Y_{k:m_2:n_2}$ can be obtained as

$$
\begin{aligned}
Y^{BP3} &= \left[E_{g^P_{(k)}}(Y^{-a^*}|data)\right]^{-\frac{1}{a^*}} \\
&= \left[\int_0^\infty y^{-a^*}\, g^P_{(k)}(y|\alpha,\lambda)\, dy\right]^{-\frac{1}{a^*}} \\
&= \left[\int_0^\infty \left(\int_0^\infty\int_0^\infty C_{k-1}\alpha\lambda\, y^{\alpha-a^*-1}\sum_{i=1}^{k} a_{i,k}\, e^{-\lambda\gamma_i\, y^\alpha}\, \pi(\alpha,\lambda|\underset{\sim}{X})\, d\alpha\, d\lambda\right)\, dy\right]^{-\frac{1}{a^*}}.
\end{aligned}
$$

As before, based on MCMC samples $\{(\alpha_l,\lambda_l);\ l=1,2,...,M\}$, the simulation estimate $\hat{Y}^{BP3}$ of $Y=Y_{k:m_2:n_2}$ will be

$$
\begin{aligned}
\hat{Y}^{BP3} &= \left[\int_0^\infty \left(\frac{1}{M}\sum_{l=1}^{M} C_{k-1}\alpha_l\lambda_l\, y^{\alpha_l-a^*-1}\sum_{i=1}^{k} a_{i,k}\, e^{-\lambda_l\gamma_i\, y^{\alpha_l}}\right)\, dy\right]^{-\frac{1}{a^*}} \\
&= \left[\frac{C_{k-1}}{M}\sum_{l=1}^{M}\alpha_l\lambda_l\,\sum_{i=1}^{k} a_{i,k}\int_0^\infty y^{\alpha_l-a^*-1}\, e^{-\lambda_l\gamma_i\, y^{\alpha_l}}\, dy\right]^{-\frac{1}{a^*}}.
\end{aligned}
$$

By making the transformation $u=\lambda_l\gamma_i y^{\alpha_l}$, and converting the integrand to the PDF of gamma distribution, the simulation estimate of $Y=Y_{k:m_2:n_2}$ becomes

$$
\begin{aligned}
\hat{Y}^{BP3} &= \left[\frac{C_{k-1}}{M}\sum_{l=1}^{M}\alpha_l\lambda_l\,\sum_{i=1}^{k} a_{i,k}\times\frac{\Gamma(1-\frac{a^*}{\alpha_l})}{\alpha_l(\lambda_l\gamma_i)^{1-\frac{a^*}{\alpha_l}}}\right]^{-\frac{1}{a^*}} \\
&= \left[\frac{C_{k-1}}{M}\sum_{l=1}^{M}\sum_{i=1}^{k} a_{i,k}\times\frac{\Gamma(1-\frac{a^*}{\alpha_l})}{(\lambda_l)^{-\frac{a^*}{\alpha_l}}(\gamma_i)^{1-\frac{a^*}{\alpha_l}}}\right]^{-\frac{1}{a^*}}. \qquad (3.36)
\end{aligned}
$$

To obtain prediction bounds on $Y=Y_{k:m_2:n_2}$, $k=1,2,\ldots,m_2$, we need the predictive distribution function of $Y=Y_{k:m_2:n_2}$, which depends on the distribution function of $Y=Y_{k:m_2:n_2}$.

Based on Eq.(3.30), the PDF of $Y=Y_{k:m_2:n_2}$, the distribution function of $Y=Y_{k:m_2:n_2}$ can be obtained as follows

$$
\begin{aligned}
G_{(k)}(y|\alpha,\lambda) &= \int_0^y g_{(k)}(z|\alpha,\lambda)\,dz \\
&= \int_0^y C_{k-1}\alpha\lambda\, z^{\alpha-1}\sum_{i=1}^{k} a_{i,k}\, e^{-\lambda\gamma_i\, z^{\alpha}}\, dz \\
&= C_{k-1}\alpha\lambda\sum_{i=1}^{k} a_{i,k}\int_y^{\infty} z^{\alpha-1}\, e^{-\lambda\gamma_i\, z^{\alpha}}\, dz.
\end{aligned}
$$

By making the transformation $u = \lambda\gamma_i z^{\alpha}$, the distribution function of $Y = Y_{k:m_2:n_2}$ becomes

$$
G_{(k)}(y|\alpha,\lambda) = C_{k-1}\sum_{i=1}^{k}\frac{a_{i,k}}{\gamma_i}(1-e^{-\lambda\gamma_i y^{\alpha}}). \tag{3.37}
$$

Under the square error loss function L_1, the predictive distribution function of $Y = Y_{k:m_2:n_2}$, can be obtained as

$$
\begin{aligned}
G^{P}_{(k)}(y|\alpha,\lambda) &= E_{posterior}\left[\,G_{(k)}(y|\alpha,\lambda)\,\right] \\
&= \int_0^{\infty}\int_0^{\infty} G_{(k)}(y|\alpha,\lambda)\,\pi(\alpha,\lambda|\underset{\sim}{X})\,d\alpha\,d\lambda \\
&= \int_0^{\infty}\int_0^{\infty} C_{k-1}\sum_{i=1}^{k}\frac{a_{i,k}}{\gamma_i}(1-e^{-\lambda\gamma_i y^{\alpha}})\,\pi(\alpha,\lambda|data)\,d\alpha\,d\lambda.
\end{aligned}
$$

Based on MCMC samples $\{(\alpha_l,\lambda_l);\ l = 1,2,...,M\}$, the simulation consistent estimator of $G^{P}_{(k)}(y|\alpha,\lambda)$ will be

$$
\hat{G}^{P}_{(k)}(y) = \frac{1}{M}\sum_{l=1}^{M}\left[C_{k-1}\sum_{i=1}^{k}\frac{a_{i,k}}{\gamma_i}(1-e^{-\lambda_l\gamma_i y^{\alpha_l}})\right].
$$

Under the absolute error loss function L_2, the simulation estimator of $G^{P}_{(k)}(y|\alpha,\lambda)$, can be obtained by using the following algorithm :

Algorithm 3

- **Step 1**
  Evaluate $G = G_{(k)}(y|\alpha, \lambda)$, Eq.(3.37), at each sample (α_l, λ_l) for $l = 1, 2, ..., M$, to get G_1 , G_2 , ... , G_M.

- **Step 2**
  Order G_1 , G_2 , ... , G_M as $G_{(1)} < G_{(2)} < ... < G_{(M)}$.

- **Step 3**
  The simulation estimator for $G^P_{(k)}(y|\alpha, \lambda)$ is given by

$$\hat{G}^P_{(k)}(y) = Median\left[G_{(1)}\ ,\ G_{(2)}\ ,\ ...\ ,\ G_{(M)}\right].$$

Under the LINEX loss function L_3 with $a^* \neq 0$, the predictive distribution function of $Y = Y_{k:m_2:n_2}$, can be obtained as

$$\begin{aligned} G^P_{(k)}(y|\alpha, \lambda) &= \left[E_{posterior}\left(G_{(k)}(y|\alpha, \lambda)\right)^{-a^*}\right]^{-\frac{1}{a^*}} \\ &= \left[\int_0^\infty\int_0^\infty \left(G_{(k)}(y|\alpha, \lambda)\right)^{-a^*} \pi(\alpha, \lambda|data)\, d\alpha\, d\lambda\right]^{-\frac{1}{a^*}} \\ &= \left[\int_0^\infty\int_0^\infty \left(C_{k-1}\sum_{i=1}^{k}\frac{a_{i,k}}{\gamma_i}(1 - e^{-\lambda\gamma_i y^\alpha})\right)^{-a^*} \pi(\alpha, \lambda|\underset{\sim}{X})\, d\alpha\, d\lambda\right]^{-\frac{1}{a^*}}. \end{aligned}$$

As before, the simulation estimator of $G^P_{(k)}(y|\alpha, \lambda)$ will be

$$\hat{G}^P_{(k)}(y) = \left[\frac{1}{M}\sum_{l=1}^{M}\left(C_{k-1}\sum_{i=1}^{k}\frac{a_{i,k}}{\gamma_i}(1 - e^{-\lambda_l\gamma_i y^{\alpha_l}}\right)^{-a*}\right]^{-\frac{1}{a^*}}.$$

Under all different loss functions L_1, L_2 and L_3, the $(1-\beta)\%$ PI of $Y_{k:m_2:n_2}$, $k = 1, 2, \ldots, m_2$, can be found by solving the non-linear equations (3.38) and (3.39) for the lower bound L and upper bound U :

$$P(Y < L|data) = \frac{\beta}{2} \Leftrightarrow \hat{G}^P_{(k)}(L) = \frac{\beta}{2}, \tag{3.38}$$

$$P(Y < U|data) = 1 - \frac{\beta}{2} \Leftrightarrow \hat{G}^P_{(k)}(U) = 1 - \frac{\beta}{2}. \tag{3.39}$$

CHAPTER 4

Statistical Inference Based on Record Data from Weibull Model

4.1 Maximum Likelihood Estimation

Let $X_{U(1)}, X_{U(2)}, ..., X_{U(n)}$ be the fist n upper record values arising from a sequence of iid Weibull variables with PDF and CDF being defined in Eq.(1.1) and Eq.(1.2), respectively. The likelihood function of this sample is

$$\begin{aligned} L(\alpha, \lambda|data) &= \prod_{i=1}^{n-1} \frac{f(x_{U(i)}|\alpha, \lambda)}{1 - F(x_{U(i)}|\alpha, \lambda)} f(x_{U(n)}|\alpha, \lambda) \\ &= \alpha^n \lambda^n e^{-\lambda x_{U(n)}^{\alpha}} \prod_{i=1}^{n} x_{U(i)}^{\alpha-1}. \end{aligned} \tag{4.1}$$

The natural logarithm of the likelihood function is

$$\ln L(\alpha, \lambda|data) = n \ln \alpha + n \ln \lambda - \lambda x_{U(n)}^{\alpha} + (\alpha - 1) \sum_{i=1}^{n} \ln x_{U(i)}, \tag{4.2}$$

where $\ln x$ denotes the natural logarithm.

By differentiating Eq.(4.2) with respect to α and λ and equating the resulting terms to zero, we obtain the following estimating equations

$$\left.\begin{aligned} \frac{\partial}{\partial \alpha} \ln L = \frac{n}{\alpha} - \lambda x_{U(n)}^{\alpha} \ln x_{U(n)} + \sum_{i=1}^{n} \ln x_{U(i)} = 0, \\ \frac{\partial}{\partial \lambda} \ln L = \frac{n}{\lambda} - x_{U(n)}^{\alpha} = 0. \end{aligned}\right\} \tag{4.3}$$

By eliminating λ in Eq.s(4.3), we obtain

$$\hat{\alpha}_{MLE} = \frac{n}{n \ln x_{U(n)} - \sum_{i=1}^{n} \ln x_{U(i)}}, \tag{4.4}$$

and then

$$\hat{\lambda}_{MLE} = n \left(x_{U(n)}\right)^{-\hat{\alpha}_{MLE}}. \tag{4.5}$$

4.2 Bayes Estimation and Credible Intervals

In this section, we estimate the unknown scale parameter λ and it's corresponding credible interval when the shape parameter α is known. Also, we use the the Gibbs sampling method to estimate the two parameters α and λ when the shape parameter α and scale parameter λ are unknown, under different loss functions and with respect to the prior(s) described in subsection [1.3.1] of chapter 1.

4.2.1 Shape Parameter Known

Based on the first upper record data $X_{U(1)}, X_{U(2)}, ..., X_{U(n)}$, and by combining the likelihood function, Eq.(4.1), and the prior density Eq.(1.3), we obtain the following theorem

Theorem 3
The conditional PDF of λ given α and data is $Gamma(a+n,\ b+x_{U(n)}^{\alpha})$ having the form

$$\pi_1(\lambda|\alpha, data) = \frac{(b + x_{U(n)}^{\alpha})^{a+n}}{\Gamma(a+n)}\ \lambda^{a+n-1}\ e^{-\lambda(b+x_{U(n)}^{\alpha})}. \tag{4.6}$$

Proof
From Eq.(4.1) and Eq.(1.3), we have

$$\begin{aligned} \pi_1(\lambda|\alpha, data) &\propto L(\alpha,\lambda)\ \pi(\lambda|a,b) \\ &\propto \lambda^n e^{-\lambda x_{U(n)}^{\alpha}}\ \lambda^{a-1} e^{-\lambda b} \\ &= \lambda^{a+n-1}\ e^{-\lambda(b+x_{U(n)}^{\alpha})}. \end{aligned}$$

Under the square error loss function L_1, the Bayes estimator $\hat{\lambda}_{B_1}$ of λ is given by

$$\hat{\lambda}_{B_1} = E_{posterior}(\lambda|\alpha, data) = \int_0^\infty \lambda\pi_1(\lambda|\alpha, data)\, d\lambda = \frac{a+n}{b + x^{\alpha}_{U(n)}}.$$

For the absolute error loss function L_2, the Bayes estimator $\hat{\lambda}_{B_2}$ of λ will be the posterior median and obtained by solving the following equation with respect to w

$$\int_0^w \pi_1(\lambda|\alpha, data)\, d\lambda = \frac{1}{2}. \tag{4.7}$$

That is, $\hat{\lambda}_{B_2}$ is the value of w satisfying Eq(4.7), or Eq.(4.8)

$$\int_w^\infty \pi_1(\lambda|\alpha, data)\, d\lambda = \frac{1}{2}. \tag{4.8}$$

Based on Eq.(4.6), Eq.(4.8) is equivalent to

$$\int_w^\infty \frac{(b + x^{\alpha}_{U(n)})^{a+n}}{\Gamma(a+n)}\, \lambda^{a+n-1}\, e^{-\lambda(b+x^{\alpha}_{U(n)})}\, d\lambda = \frac{1}{2}.$$

By making the transformation $u = \lambda(b + x^{\alpha}_{U(n)})$, we immediately obtain

$$\int_{(b+x^{\alpha}_{U(n)})w}^\infty \frac{u^{a+n-1}\, e^{-u}}{\Gamma(a+n)}\, du = \frac{1}{2},$$

or equivalently,

$$\int_{(b+x^{\alpha}_{U(n)})w}^\infty u^{a+n-1}\, e^{-u}\, du = \frac{1}{2}\Gamma(a+n).$$

Based on Eq.(3.13), the incomplete gamma function, the Bayes estimator $\hat{\lambda}_{B_2}$ of λ is the value of w satisfying the following equation

$$\Gamma\left(a+n,\ (b + x^{\alpha}_{U(n)})w\right) - \frac{1}{2}\Gamma(a+n) = 0.$$

Under the LINEX loss function L_3, for any given $a^* < a+n$, the Bayes estimator $\hat{\lambda}_{B3}$ of λ, is given by

$$\hat{\lambda}_{B3} = \left[E_{posterior}(\lambda^{-a^*}|data)\right]^{-\frac{1}{a^*}},$$

where

$$\begin{aligned} E_{posterior}(\lambda^{-a^*}|data) &= \int_0^\infty \lambda^{-a^*}\pi_1(\lambda|\alpha, data)\, d\lambda \\ &= \int_0^\infty \frac{(b+x^\alpha_{U(n)})^{a+n}}{\Gamma(a+n)}\, \lambda^{-a^*+a+n-1}\, e^{-\lambda(b+x^\alpha_{U(n)})}\, d\lambda. \end{aligned}$$

By using the PDF of gamma distribution, we have

$$E_{posterior}(\lambda^{-a^*}|data) = \frac{\Gamma(-a^*+a+n)}{\Gamma(a+n)} \left(b+x^\alpha_{U(n)}\right)^{a^*}.$$

Therefore, the Bayes estimator $\hat{\lambda}_{B3}$ of λ becomes

$$\hat{\lambda}_{B3} = \left[\frac{\Gamma(-a^*+a+n)}{\Gamma(a+n)}\right]^{-\frac{1}{a^*}} \times \frac{1}{b+x^\alpha_{U(n)}}.$$

The $(1-\beta)100\%$ credible interval of λ, (C_L, C_U), can be obtained by using the equations (3.11) and (3.12) as follows :
From Eq.(3.11), we have

$$\int_{C_L}^\infty \frac{(b+x^\alpha_{U(n)})^{a+n}}{\Gamma(a+n)}\, \lambda^{a+n-1}\, e^{-\lambda(b+x^\alpha_{U(n)})}\, d\lambda = 1-\frac{\beta}{2}.$$

By making the transformation $u = \lambda(b+x^\alpha_{U(n)})$, we immediately obtain

$$\int_{(b+x^\alpha_{U(n)})C_L}^\infty \frac{u^{a+n-1}\, e^{-u}}{\Gamma(a+n)}\, du = 1-\frac{\beta}{2},$$

which is equivalent to

$$\int_{(b+x^\alpha_{U(n)})C_L}^\infty u^{a+n-1}\, e^{-u}\, du = \left(1-\frac{\beta}{2}\right)\Gamma(a+n).$$

Based on Eq.(3.13), the lower credible interval of λ is obtained from

$$\Gamma\left(a+n,\ (b+x^{\alpha}_{U(n)})C_L\right)=\left(1-\frac{\beta}{2}\right)\Gamma(a+n). \tag{4.9}$$

Similarly from Eq.(3.12), we obtain the upper credible interval of λ from

$$\Gamma\left(a+n,\ (b+x^{\alpha}_{U(n)})C_U\right)=\frac{\beta}{2}\Gamma(a+n). \tag{4.10}$$

By solving Eq.(4.9) and Eq.(4.10) for C_L and C_U using a suitable numerical method, we obtain the lower and upper credible interval C_L and C_U, respectively.

When a is positive integer, the credible interval for λ can be obtained as follows: Since λ has $Gamma(a+n,\ b+x^{\alpha}_{U(n)})$, then a pivotal statistic $Q=2\lambda(b+x^{\alpha}_{U(n)})$ has $\chi^2_{2(a+n)}$. Hence, the $(1-\beta)100\%$ credible interval for λ is given by

$$\frac{\chi^2_{(1-\frac{\beta}{2},\ 2(a+n))}}{2(b+x^{\alpha}_{U(n)})}<\lambda<\frac{\chi^2_{(\frac{\beta}{2},\ 2(a+n))}}{2(b+x^{\alpha}_{U(n)})},$$

where $\chi^2_{(\beta,r)}$ is already defined in chapter 3.

4.2.2 Shape Parameter Unknown

In this subsection, and under different loss functions L_1, L_2 and L_3 with $a^* \neq 0$, the Bayes estimators of α and λ, or in general of $\theta = g(\alpha, \lambda)$, are obtained when both parameters α and λ are unknown. Based on the prior distributions $\pi_1(\lambda|a, b)$ and $\pi_2(\alpha)$, the posterior distribution of α and λ is defined by

$$\pi(\alpha, \lambda|data) = \frac{L(\alpha, \lambda|data) \,.\, \pi_1(\lambda|\alpha, a, b)\pi_2(\alpha)}{\int_0^\infty \int_0^\infty L(\alpha, \lambda|data) \,.\, \pi_1(\lambda|\alpha, a, b)\pi_2(\alpha)\, d\alpha\, d\lambda}. \quad (4.11)$$

Under the square loss function L_1, the Bayes estimator of $\theta = g(\alpha, \lambda)$ will be

$$\hat{\theta}_{B1} = E_{posterior}(\theta|data) = \int_0^\infty \int_0^\infty \theta\, \pi(\alpha, \lambda|data)\, d\alpha\, d\lambda.$$

For the absolute error loss function L_2, the Bayes estimator $\hat{\theta}_{B2}$ will be the median of the posterior distribution θ, *i.e.*

$$\hat{\theta}_{B2} = Med_{posterior}(\theta|data).$$

The Bayes estimator $\hat{\theta}_{B3}$ of θ, under the LINEX loss function L_3, can be obtained as

$$\hat{\theta}_{B3} = \left[E_{posterior}(\theta^{-a^*}|data)\right]^{-\frac{1}{a^*}} = \left[\int_0^\infty \int_0^\infty \theta^{-a^*}\, \pi(\alpha, \lambda|data)\, d\alpha\, d\lambda\right]^{-\frac{1}{a^*}}.$$

As in chapter 3, the Bayes estimators $\hat{\theta}_{B1}$, $\hat{\theta}_{B2}$ and $\hat{\theta}_{B3}$, under different loss functions can't be obtained in closed forms. Here we use Algorithm 1 of chapter 3 to compute the Bayes estimators above and also to construct credible intervals. To apply the Gibbs sampler method, we need the conditional distributions $\pi_1(\lambda|\alpha, data)$ and $\pi_2(\alpha|data)$. Note that $\pi_1(\lambda|\alpha, data)$ is obtained in Theorem 3, and for $\pi_2(\alpha|data)$, we state the following theorem :

Theorem 4
The conditional PDF of α given the data is given by

$$\pi_2(\alpha|data) \propto \pi_2(\alpha)\,\alpha^n \prod_{i=1}^{n} x_{U(i)}^{\alpha-1} \times \frac{1}{\left(b + x_{U(n)}^{\alpha}\right)^{a+n}}, \tag{4.12}$$

and it is log-concave.

Proof
From the posterior distribution of α and λ, Eq.(4.11), we have

$$\begin{aligned} \pi(\alpha,\lambda|data) &\propto L(\alpha,\lambda|data)\;\pi_1(\lambda|\alpha,a,b)\pi_2(\alpha) \\ &\propto \alpha^n\lambda^n e^{-\lambda x_{U(n)}^{\alpha}} \prod_{i=1}^{n} x_{U(i)}^{\alpha-1}\,.\,\lambda^{a-1}e^{-\lambda b}\pi_2(\alpha) \\ &= \pi_2(\alpha)\alpha^n \prod_{i=1}^{n} x_{U(i)}^{\alpha-1}\,.\,\lambda^{a+n-1}e^{-\lambda(b+x_{U(n)}^{\alpha})}. \end{aligned}$$

The PDF of α given data is

$$\begin{aligned} \pi_2(\alpha|data) &= \int_0^{\infty} \pi(\alpha,\lambda|data)\,d\lambda \\ &\propto \pi_2(\alpha)\alpha^n \prod_{i=1}^{n} x_{U(i)}^{\alpha-1} \int_0^{\infty} \lambda^{a+n-1}e^{-\lambda(b+x_{U(n)}^{\alpha})}\,d\lambda \\ &= \pi_2(\alpha)\alpha^n \prod_{i=1}^{n} x_{U(i)}^{\alpha-1} \times \frac{\Gamma(a+n)}{(b+x_{U(n)}^{\alpha})^{a+n}} \\ &\propto \pi_2(\alpha)\alpha^n \prod_{i=1}^{n} x_{U(i)}^{\alpha-1} \times \frac{1}{(b+x_{U(n)}^{\alpha})^{a+n}}. \end{aligned}$$

Finally, we prove the log-concavity of $\pi_2(\alpha|data)$ as follows :
Consider

$$\ln \pi_2(\alpha|data) = C + \ln \pi_2(\alpha) + n\ln\alpha + (\alpha-1)\sum_{i=1}^{n} \ln x_{U(i)} - (a+n)\ln(b + x_{U(n)}^{\alpha}),$$

where C is some constant.

Now,

$$(\ln \pi_2(\alpha|data))' = (\ln \pi_2(\alpha))' + \frac{n}{\alpha} + \sum_{i=1}^{n} \ln x_{U(i)} - (a+n)\,.\,\frac{x_{U(n)}^{\alpha}.\ln x_{U(n)}}{b + x_{U(n)}^{\alpha}}$$

$$\begin{aligned}(\ln \pi_2(\alpha|data))'' &= (\ln \pi_2(\alpha))'' - \frac{n}{\alpha^2} - (a+n)\,.\,\frac{\left[b + x_{U(n)}^{\alpha}\right]\left[x_{U(n)}^{\alpha}(\ln x_{U(n)})^2\right] - \left[x_{U(n)}^{\alpha}\ln x_{U(n)}\right]^2}{\left[b + x_{U(n)}^{\alpha}\right]^2} \\ &= (\ln \pi_2(\alpha))'' - \frac{n}{\alpha^2} - (a+n)\,.\,\frac{b\, x_{U(n)}^{\alpha}\left[\ln x_{U(n)}\right]^2}{\left[b + x_{U(n)}^{\alpha}\right]^2}.\end{aligned}$$

Since, $\pi_2(\alpha)$ is assumed log-concave, as mentioned in subsection [1.3.1] of chapter 1, we have $(\ln \pi_2(\alpha))'' < 0$, and thus $(\ln \pi_2(\alpha|data))'' < 0$, for $a, b > 0$. It follows that $\pi_2(\alpha|data)$ is log-concave density.

By using Theorems 1 and 2 and Algorithm 1 of chapter 3, we can generate MCMC samples from the posterior distribution of α and λ, Eq.(4.11), and then use these samples to obtain the Bayes estimators of $\theta = g(\alpha, \lambda)$ and the corresponding credible intervals.

4.3 Bayes Prediction

In this section, we will predict the future records based on observed records, under loss functions L_1, L_2 and L_3 with $a^* \neq 0$, when both one-sample and two-sample prediction problems are used.

Prediction problems come up naturally in several real life situations, for example, the prediction of rainfall extremes, highest water levels and sea surface or air record temperature. There has been development in this area over the past two decades. See, for example, Ahsanullah (1980) and Nagaraja (1984). Using generalized model, Bayesian prediction interval for the future generalized order statistics (including record values as a special case) was studied by Al-Hussaini and Ahmad (2003). Madi and Raqab (2004) considered the problem of Bayesian prediction of temperature records using the Pareto model.

4.3.1 One-Sample Prediction

Suppose that we observe only the first m upper records $\underset{\sim}{x} = (x_{U(1)}, x_{U(2)}, ..., x_{U(m)})$. The goal is to obtain the Bayes predictive estimator under different loss functions, as well as constructing the Bayes predictive interval for the nth future upper record $X_{U(n)}$, where $1 \leq m < n$.

By using Eq.(1.5), the posterior predictive density of $X_{U(n)}$ can be written as

$$\pi_{X_{U(n)}}(y|\underset{\sim}{x}) = \int_0^\infty \int_0^\infty f_{X_{U(n)}|\underset{\sim}{x}}(y|\alpha, \lambda)\, \pi(\alpha, \lambda|\underset{\sim}{x})\, d\alpha\, d\lambda, \;\; y > x_{U(m)},$$

where $f_{X_{U(n)}}(y|\underset{\sim}{x})$ is the conditional density function of $X_{U(n)}$ given the data $\underset{\sim}{x}$.

Using the fact that the record values satisfy Markov property, the conditional PDF of $X_{U(n)}$ given $\underset{\sim}{x}$ is just the conditional PDF of $X_{U(n)}$ given $x_{U(m)}$, *i.e.*

$$f_{X_{U(n)}|\underset{\sim}{x}}(y|\alpha, \lambda) = f_{X_{U(n)}|x_{U(m)}}(y|\alpha, \lambda).$$

Based on the equations (1.7) and (1.9), the conditional PDF of $X_{U(n)}$ given $x_{U(m)}$ can be written as follows

$$
\begin{aligned}
f_{X_{U(n)}|x_{U(m)}}(y|\alpha,\lambda) &= \frac{\left[H(y)-H(x_{U(m)})\right]^{n-m-1}}{(n-m-1)!}\,\frac{f(y|\alpha,\lambda)}{1-F(x_{U(m)}|\alpha,\lambda)},\ y > x_{U(m)} \\
&= \frac{\left[\lambda y^{\alpha}-\lambda x^{\alpha}_{U(m)}\right]^{n-m-1}}{(n-m-1)!}\,\frac{\alpha\lambda y^{\alpha-1}e^{-\lambda y^{\alpha}}}{e^{-\lambda x^{\alpha}_{U(m)}}} \\
&= \frac{\alpha\lambda^{n-m}}{(n-m-1)!}\left[y^{\alpha}-x^{\alpha}_{U(m)}\right]^{n-m-1} y^{\alpha-1}e^{-\lambda\left(y^{\alpha}-x^{\alpha}_{U(m)}\right)}.
\end{aligned}
$$

By using the binomial expansion, we immediately obtain

$$
\begin{aligned}
f_{X_{U(n)}|x_{U(m)}}(y|\alpha,\lambda) &= \frac{\alpha\lambda^{n-m}e^{\lambda x^{\alpha}_{U(m)}}}{(n-m-1)!}\left[\sum_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^{n-m-i-1}\,y^{\alpha i}\,x^{(n-m-i-1)\alpha}_{U(m)}\right] \\
&\quad\times y^{\alpha-1}e^{-\lambda y^{\alpha}} \\
&= \frac{\alpha\lambda^{n-m}e^{\lambda x^{\alpha}_{U(m)}}}{(n-m-1)!}\sum_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^{n-m-i-1}\,x^{(n-m-i-1)\alpha}_{U(m)} \\
&\quad\times y^{\alpha(i+1)-1}e^{-\lambda y^{\alpha}},\ y > x_{U(m)}. \qquad (4.13)
\end{aligned}
$$

Based on Eq.(4.13), the posterior predictive density of $X_{U(n)}$ at any point $y > x_{U(m)}$ is then

$$
\begin{aligned}
f^{P}_{X_{U(n)}|\underset{\sim}{x}}(y|\alpha,\lambda) &= E_{posterior}\left[f_{X_{U(n)}|x_{U(m)}}(y|\alpha,\lambda)\right] \\
&= \int_0^{\infty}\int_0^{\infty}\frac{\alpha\lambda^{n-m}e^{\lambda x^{\alpha}_{U(m)}}}{(n-m-1)!}\sum_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^{n-m-i-1}\,x^{(n-m-i-1)\alpha}_{U(m)} \\
&\quad\times y^{\alpha(i+1)-1}e^{-\lambda y^{\alpha}}\,\pi(\alpha,\lambda|\underset{\sim}{x})\,d\alpha d\lambda,\ y > x_{U(m)}. \qquad (4.14)
\end{aligned}
$$

Under the square error loss function L_1, the Bayes predictive estimator of $Y = X_{U(n)}$ can be obtained as

$$
\begin{aligned}
X^{BP1}_{U(n)} &= E_{f^P}(Y|\underset{\sim}{x}) \\
&= \int_{x_{U(m)}}^{\infty} y f^{P}_{X_{U(n)}|\underset{\sim}{x}}(y|\alpha,\lambda)\,dy
\end{aligned}
$$

$$= \int_{x_{U(m)}}^{\infty} \left[\int_0^{\infty}\int_0^{\infty} \frac{\alpha\lambda^{n-m} e^{\lambda x_{U(m)}^{\alpha}}}{(n-m-1)!} \sum_{i=0}^{n-m-1} \binom{n-m-1}{i} (-1)^{n-m-i-1}\, x_{U(m)}^{(n-m-i-1)\alpha} \right.$$
$$\left. \times\ y^{\alpha(i+1)} e^{-\lambda y^{\alpha}} \pi(\alpha, \lambda | \underset{\sim}{x})\, d\alpha d\lambda \right] dy.$$

As in chapter 3, using MCMC samples $\{(\alpha_j, \lambda_j);\ j = 1, 2, ..., M\}$, the simulation estimate $\hat{X}_{U(n)}^{BP1}$ of $Y = X_{U(n)}$ will be

$$\begin{aligned}
\hat{X}_{U(n)}^{BP1} &= \int_{x_{U(m)}}^{\infty} \frac{1}{M} \sum_{j=1}^{M} \frac{\alpha_j \lambda_j^{n-m} e^{\lambda_j x_{U(m)}^{\alpha_j}}}{(n-m-1)!} \sum_{i=0}^{n-m-1} \binom{n-m-1}{i} (-1)^{n-m-i-1}\, x_{U(m)}^{(n-m-i-1)\alpha_j} \\
&\quad \times y^{\alpha_j (i+1)}\, e^{-\lambda_j y^{\alpha_j}}\, dy \\
&= \frac{1}{M} \sum_{j=1}^{M} \frac{\alpha_j \lambda_j^{n-m} e^{\lambda_j x_{U(m)}^{\alpha_j}}}{(n-m-1)!} \sum_{i=0}^{n-m-1} \binom{n-m-1}{i} (-1)^{n-m-i-1}\, x_{U(m)}^{(n-m-i-1)\alpha_j} \\
&\quad \times \int_{x_{U(m)}}^{\infty} y^{\alpha_j (i+1)}\, e^{-\lambda_j y^{\alpha_j}}\, dy.
\end{aligned}$$

By making the transformation $u = \lambda_j y^{\alpha_j}$ and using the gamma distribution, the simulation estimate $\hat{X}_{U(n)}^{BP1}$ of $X_{U(n)}$ becomes

$$\begin{aligned}
\hat{X}_{U(n)}^{BP1} &= \frac{1}{M} \sum_{j=1}^{M} \frac{\alpha_j \lambda_j^{n-m} e^{\lambda_j x_{U(m)}^{\alpha_j}}}{(n-m-1)!} \sum_{i=0}^{n-m-1} \binom{n-m-1}{i} (-1)^{n-m-i-1}\, x_{U(m)}^{(n-m-i-1)\alpha_j} \\
&\quad \times \frac{\Gamma\left(\frac{1}{\alpha_j} + i + 1,\ \lambda_j x_{U(m)}^{\alpha_j}\right)}{\alpha_j \left(\lambda_j\right)^{\frac{1}{\alpha_j}+i+1}}. \\
&= \frac{1}{M} \sum_{j=1}^{M} \frac{\lambda_j^{n-m-1-\frac{1}{\alpha_j}} e^{\lambda_j x_{U(m)}^{\alpha_j}}}{(n-m-1)!} \sum_{i=0}^{n-m-1} \binom{n-m-1}{i} (-1)^{n-m-i-1}\, x_{U(m)}^{(n-m-i-1)\alpha_j} \\
&\quad \times \frac{\Gamma\left(\frac{1}{\alpha_j} + i + 1,\ \lambda_j x_{U(m)}^{\alpha_j}\right)}{\lambda_j^{i}}. \qquad (4.15)
\end{aligned}$$

It is often important to predict the first unobserved record value $X_{U(m+1)}$, the simulation estimate of the first unobserved record value can be obtained by setting $n = m + 1$ in Eq.(4.15) as follows

$$\hat{X}^{BP1}_{U(m+1)} = \frac{1}{M}\sum_{j=1}^{M} \lambda_j^{-\frac{1}{\alpha_j}}\, e^{\lambda_j x^{\alpha_j}_{U(m)}}\, \Gamma\left(\frac{1}{\alpha_j} + 1,\ \lambda_j x^{\alpha_j}_{U(m)}\right).$$

Under the absolute error loss function L_2, the corresponding Bayes predictive estimator of $X_{U(n)}$, $1 \le m < n$, denoted by $X^{BP2}_{U(n)}$ is the median of the posterior predictive density of $X_{U(n)}$, Eq.(4.14), which is obtained by solving the following equation with respect to $X^{BP2}_{U(n)}$

$$\int_{x_{U(m)}}^{X^{BP2}_{U(n)}} f^{P}_{X_{U(n)}|\underset{\sim}{x}}(y|\alpha,\lambda)\, dy = \frac{1}{2}. \tag{4.16}$$

Equation (4.16) is equivalent to

$$\int_{X^{BP2}_{U(n)}}^{\infty} f^{P}_{X_{U(n)}|\underset{\sim}{x}}(y|\alpha,\lambda)\, dy = \frac{1}{2}. \tag{4.17}$$

Based on the posterior predictive density of $X_{U(n)}$, Eq.(4.14), Eq.(4.17) is equivalent to

$$\int_{X^{BP2}_{U(n)}}^{\infty}\int_0^{\infty}\int_0^{\infty}\left[\frac{\alpha\lambda^{n-m}}{(n-m-1)!}\sum_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^{n-m-i-1}x_{U(m)}^{(n-m-i-1)\alpha}y^{\alpha(i+1)-1}e^{-\lambda\left(y^{\alpha}-x^{\alpha}_{U(m)}\right)}\right]$$
$$\times\,\pi(\alpha,\lambda|\underset{\sim}{x})\, d\alpha d\lambda\, dy = \frac{1}{2},$$

or

$$\int_{X^{BP2}_{U(n)}}^{\infty}\int_0^{\infty}\int_0^{\infty}\left[\frac{\alpha\lambda^{n-m}}{(n-m-1)!}\sum_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^{n-m-i-1}x_{U(m)}^{(n-m-i-1)\alpha}e^{\lambda x^{\alpha}_{U(m)}}\, y^{\alpha(i+1)-1}e^{-\lambda y^{\alpha}}\right]$$
$$\times\,\pi(\alpha,\lambda|\underset{\sim}{x})\, d\alpha d\lambda\, dy = \frac{1}{2}.$$

As before, based on MCMC samples $\{(\alpha_j, \lambda_j);\ j = 1, 2, ..., M\}$, the simulation estimate $\hat{X}^{BP2}_{U(n)}$ of $Y = X_{U(n)}$ can be obtained by solving, with respect to $\hat{X}^{BP2}_{U(n)}$, the following equation

$$\int\limits_{\hat{X}^{BP2}_{U(n)}}^{\infty} \frac{1}{M}\sum_{j=1}^{M} \frac{\alpha_j\lambda_j^{n-m}\, e^{\lambda_j x^{\alpha_j}_{U(m)}}}{(n-m-1)!} \sum_{i=0}^{n-m-1} \binom{n-m-1}{i} (-1)^{n-m-i-1}\, x^{(n-m-i-1)\alpha_j}_{U(m)}$$
$$\times\, y^{\alpha_j(i+1)-1}\, e^{-\lambda_j y^{\alpha_j}}\, dy = \frac{1}{2},$$

or

$$\frac{1}{M}\sum_{j=1}^{M} \frac{\alpha_j\lambda_j^{n-m}\, e^{\lambda_j x^{\alpha_j}_{U(m)}}}{(n-m-1)!} \sum_{i=0}^{n-m-1} \binom{n-m-1}{i} (-1)^{n-m-i-1}\, x^{(n-m-i-1)\alpha_j}_{U(m)}$$
$$\times \int\limits_{\hat{X}^{BP2}_{U(n)}}^{\infty} \left(y^{\alpha_j(i+1)-1}\, e^{-\lambda_j y^{\alpha_j}}\right) dy = \frac{1}{2}.$$

By making the transformation $u = \lambda_j y^{\alpha_j}$, we have

$$\frac{1}{M}\sum_{j=1}^{M} \frac{\alpha_j\lambda_j^{n-m}\, e^{\lambda_j x^{\alpha_j}_{U(m)}}}{(n-m-1)!} \sum_{i=0}^{n-m-1} \binom{n-m-1}{i} (-1)^{n-m-i-1}\, x^{(n-m-i-1)\alpha_j}_{U(m)}$$
$$\times \left[\frac{1}{\alpha_j\lambda_j^{i+1}} \int\limits_{\lambda_j(\hat{X}^{BP2}_{U(n)})^{\alpha_j}}^{\infty} u^i\, e^{-u} du\right] = \frac{1}{2},$$

or

$$\frac{1}{M}\sum_{j=1}^{M} \frac{e^{\lambda_j x^{\alpha_j}_{U(m)}}}{(n-m-1)!} \sum_{i=0}^{n-m-1} \binom{n-m-1}{i} (-\lambda_j)^{n-m-i-1}\, x^{(n-m-i-1)\alpha_j}_{U(m)}$$
$$\times \left[\int\limits_{\lambda_j(\hat{X}^{BP2}_{U(n)})^{\alpha_j}}^{\infty} u^i\, e^{-u} du\right] = \frac{1}{2}.$$

By using Eq.(3.13), the above equation can be written as

$$\frac{1}{M}\sum_{j=1}^{M} \frac{e^{\lambda_j x^{\alpha_j}_{U(m)}}}{(n-m-1)!} \sum_{i=0}^{n-m-1} \binom{n-m-1}{i} (-\lambda_j)^{n-m-i-1}\, x^{(n-m-i-1)\alpha_j}_{U(m)}$$
$$\times\, \Gamma\left(i+1,\ \left(\lambda_j(\hat{X}^{BP2}_{U(n)})^{\alpha_j}\right)\right) = \frac{1}{2}. \qquad (4.18)$$

Therefore, the simulation estimate $\hat{X}^{BP2}_{U(n)}$ of $X_{U(n)}$ is the solution of Eq.(4.18), with respect to $\hat{X}^{BP2}_{U(n)}$.

In particular, the simulation estimate $\hat{X}^{BP2}_{U(m+1)}$ of $X_{U(m+1)}$ can be obtained by setting $n = m+1$ in Eq.(4.18) and solving, with respect to $\hat{X}^{BP2}_{U(m+1)}$, the following equation

$$\frac{1}{M}\sum_{j=1}^{M} e^{\lambda_j x^{\alpha_j}_{U(m)}}\,\Gamma(1,\ \lambda_j(\hat{X}^{BP2}_{U(m+1)})^{\alpha_j}) \;=\; \frac{1}{2},$$

or

$$\sum_{j=1}^{M} e^{\lambda_j x^{\alpha_j}_{U(m)}}\,\Gamma(1,\ \lambda_j(\hat{X}^{BP2}_{U(m+1)})^{\alpha_j}) \;=\; \frac{M}{2}.$$

Under the LINEX loss function L_3, the corresponding Bayes predictive estimator $X^{BP3}_{U(n)}$ of $Y = X_{U(n)}$ can be obtained as

$$\begin{aligned}
X^{BP3}_{U(n)} &= \Big[E_{f^P}(Y^{-a^*}|\underset{\sim}{x})\Big]^{-\frac{1}{a^*}} \\
&= \left[\int_{x_{U(m)}}^{\infty} y^{-a^*} f^P_{X_{U(n)}|\underset{\sim}{x}}(y|\alpha,\lambda)\,dy\right]^{-\frac{1}{a^*}} \\
&= \Bigg[\int_{x_{U(m)}}^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{\alpha\lambda^{n-m}}{(n-m-1)!}\sum_{i=0}^{n-m-1}\binom{n-m-1}{i}(-1)^{n-m-i-1}\,x^{(n-m-i-1)\alpha}_{U(m)} \\
&\qquad \times\, y^{\alpha(i+1)-a^*-1}\,e^{-\lambda(y^{\alpha}-x^{\alpha}_{U(m)})}\,\pi(\alpha,\lambda|\underset{\sim}{x})\,d\alpha d\lambda\,dy\Bigg]^{-\frac{1}{a^*}}.
\end{aligned}$$

Based on MCMC samples $\{(\alpha_j,\lambda_j);\ j = 1,2,...,M\}$, the simulation estimate $\hat{X}^{BP3}_{U(n)}$ of $X_{U(n)}$ will be

$$\hat{X}^{BP3}_{U(n)} = \left[\int\limits_{x_{U(m)}}^{\infty} \frac{1}{M}\sum_{j=1}^{M} \frac{\alpha_j \lambda_j^{n-m}}{(n-m-1)!} \sum_{i=0}^{n-m-1} \binom{n-m-1}{i} (-1)^{n-m-i-1}\, x_{U(m)}^{(n-m-i-1)\alpha_j} \right.$$

$$\left. \times\ y^{\alpha_j(i+1)-a^*-1}\, e^{-\lambda_j\left(y^{\alpha_j}-x_{U(m)}^{\alpha_j}\right)}\, dy \right]^{-\frac{1}{a^*}},$$

$$= \left[\frac{1}{M}\sum_{j=1}^{M} \frac{\alpha_j \lambda_j^{n-m}}{(n-m-1)!} \sum_{i=0}^{n-m-1} \binom{n-m-1}{i} (-1)^{n-m-i-1}\, x_{U(m)}^{(n-m-i-1)\alpha_j}\, e^{\lambda_j x_{U(m)}^{\alpha_j}} \right.$$

$$\left. \times \int\limits_{x_{U(m)}}^{\infty} \left(y^{\alpha_j(i+1)-a^*-1}\, e^{-\lambda_j y^{\alpha_j}}\right) dy \right]^{-\frac{1}{a^*}}.$$

By making the transformation $u = \lambda_j y^{\alpha_j}$ and using the gamma distribution and Eq.(3.13), the simulation estimate of $X_{U(n)}$ becomes

$$\hat{X}^{BP3}_{U(n)} = \left[\frac{1}{M}\sum_{j=1}^{M} \frac{\alpha_j \lambda_j^{n-m}\, e^{\lambda_j x_{U(m)}^{\alpha_j}}}{(n-m-1)!} \sum_{i=0}^{n-m-1} \binom{n-m-1}{i} (-1)^{n-m-i-1}\, x_{U(m)}^{(n-m-i-1)\alpha_j} \right.$$

$$\left. \times\ \frac{\Gamma\left(i-\frac{a^*}{\alpha_j}+1,\ \lambda_j x_{U(m)}^{\alpha_j}\right)}{\alpha_j \lambda_j^{i-\frac{a^*}{\alpha_j}+1}} \right]^{-\frac{1}{a^*}},$$

$$= \left[\frac{1}{M}\sum_{j=1}^{M} \frac{\lambda_j^{n-m-1+\frac{a^*}{\alpha_j}}\, e^{\lambda_j x_{U(m)}^{\alpha_j}}}{(n-m-1)!} \sum_{i=0}^{n-m-1} \binom{n-m-1}{i} (-1)^{n-m-i-1}\, x_{U(m)}^{(n-m-i-1)\alpha_j} \right.$$

$$\left. \times\ \frac{\Gamma\left(i-\frac{a^*}{\alpha_j}+1,\ \lambda_j x_{U(m)}^{\alpha_j}\right)}{\lambda^{i}} \right]^{-\frac{1}{a^*}}. \tag{4.19}$$

In particular, the simulation estimate $\hat{X}^{BP3}_{U(m+1)}$ of $X_{U(m+1)}$ can be obtained by setting $n = m+1$ in Eq.(4.19) as follows

$$\hat{X}^{BP3}_{U(m+1)} = \left[\frac{1}{M}\sum_{j=1}^{M} \lambda_j^{\frac{a^*}{\alpha_j}}\, e^{\lambda_j x_{U(m)}^{\alpha_j}}\, \Gamma\left(1-\frac{a^*}{\alpha_j},\ \lambda_j x_{U(m)}^{\alpha_j}\right)\right]^{-\frac{1}{a^*}}.$$

To obtain prediction bounds on $Y = X_{U(n)}$, $1 \leq m < n$, under different loss functions, we need to find the predictive survival function of $Y = X_{U(n)}$ at any point $y > x_{U(m)}$. The predictive survival function is defined by

$$\begin{aligned} S^{P}_{X_{U(n)}|\underset{\sim}{x}}(y|\alpha,\lambda) &= E_{posterior}\left(S_{X_{U(n)}|\underset{\sim}{x}}(y|\alpha,\lambda)\right) \\ &= \int_0^\infty\int_0^\infty S_{X_{U(n)}|\underset{\sim}{x}}(y|\alpha,\lambda)\pi(\alpha,\lambda|\underset{\sim}{x})\,d\alpha\,d\lambda, \end{aligned} \tag{4.20}$$

where $S_{X_{U(n)}|\underset{\sim}{x}}$ is the survival function of $Y = X_{U(n)}$.
Using the Markovian property of the record order statistics, we have

$$S_{X_{U(n)}|\underset{\sim}{x}}(y|\alpha,\lambda) = S_{X_{U(n)}|x_{U(m)}}(y|\alpha,\lambda).$$

Now

$$\begin{aligned} S_{X_{U(n)}|x_{U(m)}}(y|\alpha,\lambda) &= P(Y > y|x_{U(m)}) \\ &= \int_y^\infty f_{X_{U(n)}|x_{U(m)}}(z|\alpha,\lambda)\,dz \\ &= \int_y^\infty \frac{\left[H(z) - H(x_{U(m)})\right]^{n-m-1}}{(n-m-1)!}\,\frac{f(z|\alpha,\lambda)}{1 - F(x_{U(m)}|\alpha,\lambda)}\,dz. \end{aligned}$$

By making the transformation $v = H(z) - H(x_{U(m)})$, we have

$$H(z) = v + H(x_{U(m)}), \text{ and } 1 - F(z) = e^{-v}(1 - F(x_{U(m)})).$$

This in turn, the survival function becomes

$$S_{X_{U(n)}|x_{U(m)}}(y|\alpha,\lambda) = \int_{H(y)-H(x_{U(m)})}^\infty \frac{v^{n-m-1}\,e^{-v}}{(n-m-1)!}\,dv.$$

By using the relation between the incomplete gamma function and sum of poisson probabilities

$$\int_x^\infty \frac{u^{k-1}\,e^{-u}}{\Gamma(k)}\,du = \sum_{j=0}^{k-1}\frac{e^{-x}\,x^j}{j!}, \tag{4.21}$$

the survival function becomes

$$\begin{aligned} S_{X_{U(n)}|x_{U(m)}}(y|\alpha,\lambda) &= \sum_{j=0}^{n-m-1} \frac{e^{-\left[H(y)-H(x_{U(m)})\right]} \left[H(y)-H(x_{U(m)})\right]^j}{j!} \\ &= \sum_{j=0}^{n-m-1} \frac{\left[\frac{1-F(y|\alpha,\lambda)}{1-F(x_{U(m)}|\alpha,\lambda)}\right] \left[-\ln\left(\frac{1-F(y|\alpha,\lambda)}{1-F(x_{U(m)}|\alpha,\lambda)}\right)\right]^j}{j!}. \end{aligned}$$

Since

$$\frac{1-F(y|\alpha,\lambda)}{1-F(x_{U(m)}|\alpha,\lambda)} = \frac{e^{-\lambda y^\alpha}}{e^{-\lambda x_{U(m)}^\alpha}} = e^{-\lambda(y^\alpha - x_{U(m)}^\alpha)},$$

then the survival function becomes

$$S_{X_{U(n)}|x_{U(m)}}(y|\alpha,\lambda) = \sum_{j=0}^{n-m-1} \frac{e^{-\lambda(y^\alpha - x_{U(m)}^\alpha)} \left[\lambda(y^\alpha - x_{U(m)}^\alpha)\right]^j}{j!}. \tag{4.22}$$

The predictive survival function for $Y = X_{U(n)}$, Eq.(4.20), becomes

$$S^P_{X_{U(n)}|\underset{\sim}{x}}(y|\alpha,\lambda) = \int_0^\infty \int_0^\infty \left[\sum_{j=0}^{n-m-1} \frac{e^{-\lambda(y^\alpha - x_{U(m)}^\alpha)} \left[\lambda(y^\alpha - x_{U(m)}^\alpha)\right]^j}{j!}\right] \pi(\alpha,\lambda|\underset{\sim}{x})\, d\alpha\, d\lambda. \tag{4.23}$$

Notice that Eq.(4.23) can't be expressed in closed form and hence can't be evaluated analytically. By using the MCMC samples $\{(\alpha_i,\lambda_i);\ i=1,2,...,M\}$ obtained by using the Gibbs sampler method, and under the square error loss function L_1, the simulation consistent estimator of the predictive survival function for $X_{U(n)}$ will be

$$\hat{S}^P_{X_{U(n)}|\underset{\sim}{x}}(y) = \frac{1}{M}\sum_{i=1}^{M}\left[\sum_{j=0}^{n-m-1} \frac{e^{-\lambda_i(y^{\alpha_i} - x_{U(m)}^{\alpha_i})} \left[\lambda_i(y^{\alpha_i} - x_{U(m)}^{\alpha_i})\right]^j}{j!}\right].$$

Under the absolute error loss function L_2, the simulation estimator of the predictive survival function for $X_{U(n)}$, can be obtained by using the following algorithm :

Algorithm 4

- **Step 1**
  Evaluate $S = S_{X_{U(n)}|x_{U(m)}}(y|\alpha,\lambda)$, Eq.(4.22), at each sample (α_i,λ_i) for $i = 1,2,...,M$, to get S_1 , S_2 , ... , S_M

- **Step 2**
  Order S_1 , S_2 , ... , S_M as $S_{(1)} < S_{(2)} < ... < S_{(M)}$

- **Step 3**
  The simulation estimator of the predictive survival function for $X_{U(n)}$ is given by

$$\hat{S}^P_{X_{U(n)}|\underset{\sim}{x}}(y) = Median\left[S_{(1)}\ ,\ S_{(2)}\ ,\ ...\ ,\ S_{(M)}\right]$$

Under the LINEX loss function L_3, the simulation estimator of the predictive survival function for $X_{U(n)}$, can be obtained as

$$\hat{S}^P_{X_{U(n)}|\underset{\sim}{x}}(y) = \left[\frac{1}{M}\sum_{i=1}^{M}\left(\sum_{j=0}^{n-m-1}\frac{e^{-\lambda_i(y^{\alpha_i}-x^{\alpha_i}_{U(m)})}\,[\lambda_i(y^{\alpha_i}-x^{\alpha_i}_{U(m)})]^j}{j!}\right)^{-a^*}\right]^{-\frac{1}{a^*}}.$$

A $(1-\beta)\%$ PI for $X_{U(n)}$, $1 \leq m < n$, can be obtained by solving the non-linear equations (4.24) and (4.25) for the lower bound L and upper bound U :

$$P(X_{U(n)} > L|\underset{\sim}{x}) = 1-\frac{\beta}{2} \;\Leftrightarrow\; \hat{S}^P_{X_{U(n)}|\underset{\sim}{x}}(L) = 1-\frac{\beta}{2}, \tag{4.24}$$

$$P(X_{U(n)} > U|\underset{\sim}{x}) = \frac{\beta}{2} \;\Leftrightarrow\; \hat{S}^P_{X_{U(n)}|\underset{\sim}{x}}(U) = \frac{\beta}{2}. \tag{4.25}$$

We need to apply a suitable numerical method to solve these non-linear equations as they can't be solved analytically. In particular, a $(1-\beta)\%$ PI for $X_{U(m+1)}$ can be obtained by solving the equations (4.24) and (4.25) when $n = m+1$.

4.3.2 Two-Sample Prediction

Let $X_{U(1)}$, $X_{U(2)}$, ... , $X_{U(m)}$ be the first m observed records from a sequence with WE(α,λ). Let $Y_{U(1)}$, $Y_{U(2)}$, ... , $Y_{U(n)}$ be the first n record values from another independent sequence sample from the same distribution.

Based on the observed record sample, we are interested in the predicting of the kth upper record value $Y_{U(k)}$, $1 \leq k \leq n$, of the future sequence, and obtaining prediction interval of $Y_{U(k)}$.
The PDF of the kth upper record value $Y_{U(k)}$ is given by [see Eq.(1.7) and Eq.(1.8)]

$$\begin{aligned} g_{(k)}(y|\alpha,\lambda) &= \frac{[-\ln(1-F(y|\alpha,\lambda))]^{k-1}}{\Gamma(k)} f(y|\alpha,\lambda) \\ &= \frac{[-\ln(e^{-\lambda y^{\alpha}})]^{k-1}}{\Gamma(k)}\, \alpha\lambda\, y^{\alpha-1}\, e^{-\lambda y^{\alpha}} \\ &= \frac{\alpha\lambda^{k}}{\Gamma(k)}\, y^{\alpha k-1}\, e^{-\lambda y^{\alpha}}. \end{aligned} \tag{4.26}$$

To obtain the Bayes predictive estimator of $Y = Y_{U(k)}$, $1 \leq k \leq n$, under different loss functions, we need the posterior predictive density of $Y_{U(k)}$.
The posterior predictive density of $Y_{U(k)}$ is denoted by $g^{P}_{(k)}(y|\alpha,\lambda)$ and given by

$$\begin{aligned} g^{P}_{(k)}(y|\alpha,\lambda) &= E_{posterior}\left[g_{(k)}(y|\alpha,\lambda)\right] \\ &= \int_0^{\infty}\int_0^{\infty} g_{(k)}(y|\alpha,\lambda)\, \pi(\alpha,\lambda|data)\, d\alpha\, d\lambda \\ &= \int_0^{\infty}\int_0^{\infty} \frac{\alpha\lambda^{k}}{\Gamma(k)}\, y^{\alpha k-1}\, e^{-\lambda y^{\alpha}}\, \pi(\alpha,\lambda|data)\, d\alpha\, d\lambda. \end{aligned} \tag{4.27}$$

Under the square error loss function L_1, the Bayes predictive estimator of $Y = Y_{U(k)}$ can be obtained as

$$\begin{aligned} Y^{BP1}_{U(k)} &= E_{g^{P}_{(k)}}(Y|data) \\ &= \int_0^{\infty} y\, g^{P}_{(k)}(y|\alpha,\lambda)\, dy \\ &= \int_0^{\infty}\left[\int_0^{\infty}\int_0^{\infty} \frac{\alpha\lambda^{k}}{\Gamma(k)}\, y^{\alpha k}\, e^{-\lambda y^{\alpha}}\, \pi(\alpha,\lambda|data)\, d\alpha\, d\lambda\right] dy. \end{aligned}$$

As before, based on MCMC samples $\{(\alpha_i,\lambda_i);\ i = 1,2,...,M\}$, the simulation estimate $\hat{Y}^{BP1}_{U(k)}$of $Y = Y_{U(k)}$ will be

$$\begin{aligned} \hat{Y}^{BP1}_{U(k)} &= \int_0^{\infty}\left[\frac{1}{M}\sum_{i=1}^{M} \frac{\alpha_i\lambda_i^{k}}{\Gamma(k)}\, y^{\alpha_i k}\, e^{-\lambda_i y^{\alpha_i}}\right] dy \\ &= \frac{1}{M}\sum_{i=1}^{M} \frac{\alpha_i\lambda_i^{k}}{\Gamma(k)} \int_0^{\infty} y^{\alpha_i k}\, e^{-\lambda_i y^{\alpha_i}}\, dy. \end{aligned}$$

By making the transformation $u = \lambda_i y^{\alpha_i}$ and using the gamma distribution, the simulation estimate of $Y_{U(k)}^{BP1}$ becomes

$$\begin{aligned}\hat{Y}_{U(k)}^{BP1} &= \frac{1}{M}\sum_{i=1}^{M}\frac{\alpha_i\lambda_i^k}{\Gamma(k)}\frac{\Gamma(k+\frac{1}{\alpha_i})}{\alpha_i\lambda_i^{k+\frac{1}{\alpha_i}}}\\ &= \frac{1}{M}\sum_{i=1}^{M}\frac{\Gamma(k+\frac{1}{\alpha_i})}{\lambda_i^{\frac{1}{\alpha_i}}\Gamma(k)} \qquad (4.28)\end{aligned}$$

Under the absolute error loss function L_2, the corresponding Bayes predictive estimator of $Y_{U(k)}$ denoted by $Y_{U(k)}^{BP2}$, is the median of the posterior predictive density of $Y_{U(k)}$, Eq.(4.27), which is obtained by solving the following equation with respect to $Y_{U(k)}^{BP2}$

$$\int_0^{Y_{U(k)}^{BP2}} g_{(k)}^{P}(y|\alpha,\lambda)\,dy = \frac{1}{2}, \qquad (4.29)$$

or

$$\int_{Y_{U(k)}^{BP2}}^{\infty} g_{(k)}^{P}(y|\alpha,\lambda)\,dy = \frac{1}{2}. \qquad (4.30)$$

Based on Eq.(4.27), Eq.(4.30) is equivalent to

$$\int_{Y_{U(k)}^{BP2}}^{\infty}\left[\int_0^{\infty}\int_0^{\infty}\frac{\alpha\lambda^k}{\Gamma(k)}\,y^{\alpha k-1}\,e^{-\lambda y^{\alpha}}\,\pi(\alpha,\lambda|data)\,d\alpha\,d\lambda\right]dy = \frac{1}{2}.$$

Based on MCMC samples $\{(\alpha_i,\lambda_i);\ i=1,2,...,M\}$, the simulation estimate $\hat{Y}_{U(k)}^{BP2}$ of $Y_{U(k)}^{BP2}$ can be obtained by solving, with respect to $\hat{Y}_{U(k)}^{BP2}$, the following equation

$$\int_{\hat{Y}_{U(k)}^{BP2}}^{\infty}\left[\frac{1}{M}\sum_{i=1}^{M}\frac{\alpha_i\lambda_i^k}{\Gamma(k)}\,y^{\alpha_i k-1}\,e^{-\lambda_i y^{\alpha_i}}\right]dy = \frac{1}{2},$$

which is equivalent to solving

$$\frac{1}{M}\sum_{i=1}^{M}\left[\frac{\alpha_i\lambda_i^k}{\Gamma(k)}\int_{\hat{Y}_{U(k)}^{BP2}}^{\infty} y^{\alpha_i k-1}\,e^{-\lambda_i y^{\alpha_i}}\,dy\right] = \frac{1}{2}. \qquad (4.31)$$

By making the transformation $u = \lambda_i y^{\alpha_i}$, Eq.(4.31) is equivalent to

$$\frac{1}{M}\sum_{i=1}^{M}\left[\frac{\alpha_i\lambda_i^k}{\Gamma(k)}\left(\frac{1}{\alpha_i\lambda_i^k}\int\limits_{\lambda_i(\hat{Y}_{U(k)}^{BP2})^{\alpha_i}}^{\infty} u^{k-1}\, e^{-u}\, du\right)\right] = \frac{1}{2}. \tag{4.32}$$

Based on Eq.(3.13), Eq.(4.32) becomes

$$\frac{1}{M}\sum_{i=1}^{M}\left[\frac{\Gamma\left(k,\ \lambda_i(\hat{Y}_{U(k)}^{BP2})^{\alpha_i}\right)}{\Gamma(k)}\right] = \frac{1}{2}. \tag{4.33}$$

The simulation estimate $\hat{Y}_{U(k)}^{BP2}$ of $Y_{U(k)}^{BP2}$, under the absolute error loss function L_2, can be obtained by solving Eq.(4.33) with respect to $\hat{Y}_{U(k)}^{BP2}$.
Under the LINEX loss function L_3, the corresponding Bayes predictive estimator $Y_{U(k)}^{BP3}$ of $Y = Y_{U(k)}$, for $1 \leq k \leq n$, can be obtained as

$$\begin{aligned}
Y_{U(k)}^{BP3} &= \left[E_{g_{(k)}^P}(Y^{-a^*}|data)\right]^{-\frac{1}{a^*}} \\
&= \left[\int\limits_0^{\infty} y^{-a^*}\, g_{(k)}^P(y|\alpha,\lambda)\, dy\right]^{-\frac{1}{a^*}} \\
&= \left[\int\limits_0^{\infty}\left(\int\limits_0^{\infty}\int\limits_0^{\infty}\frac{\alpha\lambda^k}{\Gamma(k)}\, y^{\alpha k - a^* - 1}\, e^{-\lambda y^{\alpha}}\, \pi(\alpha,\lambda|data)\, d\alpha\, d\lambda\right)\, dy\right]^{-\frac{1}{a^*}}.
\end{aligned}$$

Based on MCMC samples $\{(\alpha_i, \lambda_i);\ i = 1, 2, ..., M\}$, the simulation estimate $\hat{Y}_{U(k)}^{BP3}$ of $Y_{U(k)}^{BP3}$ will be

$$\begin{aligned}
\hat{Y}_{U(k)}^{BP3} &= \left[\int\limits_0^{\infty}\left(\frac{1}{M}\sum_{i=1}^{M}\frac{\alpha_i\lambda_i^k}{\Gamma(k)}\, y^{\alpha_i k - a^* - 1}\, e^{-\lambda_i y^{\alpha_i}}\right)\, dy\right]^{-\frac{1}{a^*}} \\
&= \left[\frac{1}{M}\sum_{i=1}^{M}\left(\frac{\alpha_i\lambda_i^k}{\Gamma(k)}\int\limits_0^{\infty} y^{\alpha_i k - a^* - 1}\, e^{-\lambda_i y^{\alpha_i}}\, dy\right)\right]^{-\frac{1}{a^*}}.
\end{aligned}$$

By making the transformation $u = \lambda_i y^{\alpha_i}$, the simulation estimate $\hat{Y}_{U(k)}^{BP3}$ of $Y_{U(k)}^{BP3}$ becomes

$$\begin{aligned}
\hat{Y}_{U(k)}^{BP3} &= \left[\frac{1}{M}\sum_{i=1}^{M}\left(\frac{\alpha_i\lambda_i^k}{\Gamma(k)}\frac{\Gamma(k-\frac{a^*}{\alpha_i})}{\alpha_i\lambda_i^{k-\frac{a^*}{\alpha_i}}}\right)\right]^{-\frac{1}{a^*}} \\
&= \left[\frac{1}{M}\sum_{i=1}^{M}\frac{\lambda_i^{\frac{a^*}{\alpha_i}}\Gamma\left(k-\frac{a^*}{\alpha_i}\right)}{\Gamma(k)}\right]^{-\frac{1}{a^*}}. \qquad (4.34)
\end{aligned}$$

To obtain prediction bounds on $Y = Y_{U(k)}$, for $1 \leq k \leq n$, we need the predictive distribution function of $Y = Y_{U(k)}$, which depends on the distribution function of $Y = Y_{U(k)}$.
Based on Eq.(4.26), the PDF of $Y_{U(k)}$, the distribution function of $Y_{U(k)}$ can be obtained as follows

$$\begin{aligned}
G_{(k)}(y|\alpha,\lambda) &= \int_0^y g_{(k)}(z|\alpha,\lambda)\,dz \\
&= 1-\int_y^\infty g_{(k)}(z|\alpha,\lambda)\,dz \\
&= 1-\int_y^\infty \frac{\alpha\lambda^k}{\Gamma(k)}\,z^{\alpha k-1}\,e^{-\lambda z^\alpha}\,dz \\
&= 1-\frac{\alpha\lambda^k}{\Gamma(k)}\int_y^\infty z^{\alpha k-1}\,e^{-\lambda z^\alpha}\,dz.
\end{aligned}$$

By making the transformation $u = \lambda z^\alpha$, we have

$$G_{(k)}(y|\alpha,\lambda) = 1-\frac{\alpha\lambda^k}{\Gamma(k)}\left[\frac{1}{\alpha\lambda^k}\int_{\lambda y^\alpha}^\infty u^{k-1}\,e^{-u}\,du\right].$$

Based on Eq.(3.13), the distribution function of $Y_{U(k)}$ becomes

$$G_{(k)}(y|\alpha,\lambda) = 1-\frac{\Gamma(k,\ \lambda y^\alpha)}{\Gamma(k)}. \qquad (4.35)$$

Under the square error loss function L_1, the predictive distribution function of $Y_{U(k)}$ can be obtained as

$$\begin{aligned} G^P_{(k)}(y|\alpha,\lambda) &= E_{posterior}\left[\, G_{(k)}(y|\alpha,\lambda)\,\right] \\ &= \int_0^\infty\int_0^\infty G_{(k)}(y|\alpha,\lambda)\,\pi(\alpha,\lambda|data)\,d\alpha\,d\lambda \\ &= \int_0^\infty\int_0^\infty \left[1-\frac{\Gamma(k,\ \lambda y^{\alpha})}{\Gamma(k)}\right]\,\pi(\alpha,\lambda|data)\,d\alpha\,d\lambda. \end{aligned}$$

Based on MCMC samples $\{(\alpha_i,\lambda_i);\ i=1,2,...,M\}$ obtained by the Gibbs sampler method, the simulation estimator of $G^P_{(k)}(y|\alpha,\lambda)$ will be

$$\hat{G}^P_{(k)}(y) = \frac{1}{M}\sum_{i=1}^{M}\left[1-\frac{\Gamma(k,\ \lambda_i y^{\alpha_i})}{\Gamma(k)}\right]$$

Under the absolute error loss function L_2, the simulation estimator of $G^P_{(k)}(y|\alpha,\lambda)$ can be obtained by using the following algorithm :

Algorithm 5

- **Step 1**
  Evaluate $G = G_{(k)}(y|\alpha,\lambda)$, Eq.(4.35), at each sample (α_i,λ_i) for $i=1,2,...,M$, to get $G_1\ ,\ G_2\ ,\ ...\ ,\ G_M$

- **Step 2**
  Order $G_1\ ,\ G_2\ ,\ ...\ ,\ G_M$ as $G_{(1)}\ <\ G_{(2)}\ <\ ...\ <\ G_{(M)}$

- **Step 3**
  The simulation estimator for $G^P_{(k)}(y|\alpha,\lambda)$ is given by

$$\hat{G}^P_{(k)}(y) = Median\left[G_{(1)}\ ,\ G_{(2)}\ ,\ ...\ ,\ G_{(M)}\right]$$

Under the LINEX loss function L_3, the predictive distribution function of $Y_{U(k)}$ can be obtained as

$$\begin{aligned} G^P_{(k)}(y|\alpha,\lambda) &= \left[E_{posterior}\left(G_{(k)}(y|\alpha,\lambda)\right)^{-a^*}\right]^{-\frac{1}{a^*}} \\ &= \left[\int_0^\infty\int_0^\infty \left(G_{(k)}(y|\alpha,\lambda)\right)^{-a^*}\pi(\alpha,\lambda|data)\,d\alpha\,d\lambda\right]^{-\frac{1}{a^*}} \\ &= \left[\int_0^\infty\int_0^\infty \left(1-\frac{\Gamma(k,\ \lambda y^\alpha)}{\Gamma(k)}\right)^{-a^*}\pi(\alpha,\lambda|data)\,d\alpha\,d\lambda\right]^{-\frac{1}{a^*}}. \end{aligned}$$

Based on MCMC samples $\{(\alpha_i,\lambda_i);\ i=1,2,...,M\}$, the simulation estimator of $G^P_{(k)}(y|\alpha,\lambda)$ will be

$$\hat{G}^P_{(k)}(y) = \left[\frac{1}{M}\sum_{i=1}^{M}\left(1-\frac{\Gamma(k,\ \lambda_i y^{\alpha_i})}{\Gamma(k)}\right)^{-a*}\right]^{-\frac{1}{a^*}}.$$

Under all different loss functions L_1, L_2 and L_3, the $(1-\beta)\%$ PI for $Y=Y_{U(k)}$, $1\le k\le n$, can be obtained by solving the non-linear equations (4.36) and (4.37) for the lower bound L and upper bound U :

$$P(Y<L|data)=\frac{\beta}{2}\ \Leftrightarrow\ \hat{G}^P_{(k)}(L)=\frac{\beta}{2}, \tag{4.36}$$

$$P(Y<U|data)=1-\frac{\beta}{2}\ \Leftrightarrow\ \hat{G}^P_{(k)}(U)=1-\frac{\beta}{2}. \tag{4.37}$$

As before, we need a suitable numerical method to solve these non-linear equations.

CHAPTER 5

Simulation Study

The equations for estimation and prediction obtained in chapters 3 and 4 based on different loss functions described in subsection [1.3.1] can't be solved analytically. For this, we use Mathematica Package V7 to solve these equations.
In this chapter we conduct a simulation study to examine the behavior of the maximum likelihood estimator and the Bayes estimators as well as the different Bayes predictive estimators, based on progressive type II censoring data and record data. In both cases Bayes estimation and prediction, we have assumed $\alpha = 2$, $\lambda = 1$ to generate progressive type II censored data and record data. The progressive type II censored data were generated by using the algorithm proposed by Balakrishnan and Aggarwala (2000). The first n observed records were generated by using the transformation:

$$X_{U(k)} = \left(\frac{\sum_{i=1}^{k} e(i)}{\lambda} \right)^{\frac{1}{\alpha}}, k = 1, 2, ..., n,$$

where $\{e(i), i \geq 0\}$ is a sequence of i.i.d Exp(1) [see Arnold et al. (1998), p.20].
For both types of data (progressive and record data) we have computed the Bayes estimators, for the Weibull parameters α and λ, with respect to different loss functions: square error (Sq. err.) (L_1), absolute error (Abs. err.) (L_2) and LINEX (L_3) with different choices of $a^* : 0.1, 1, 5$. To compute the different Bayes estimators we have assumed $\pi_2(\alpha)$, the prior of α, has gamma density function with the shape and scale parameters c and d, respectively. For the computations of Bayes estimators, we consider two types of prior for both α and λ: first prior is the non-informative prior, *i.e* $a = b = c = d = 0$, we call this prior as Prior 0, second prior is the informative prior, namely $a = b = 1$, $c = 2, d = 1$, we call this prior as Prior 1. We have computed the MSEs for the different Bayes estimators based on 1000 replications.
The credible intervals for the Weibull parameters α and λ are computed.

For the computations of prediction, we consider only Prior 1 and loss functions: Sq. err. (L_1), Abs. err. (L_2) and LINEX (L_3) with different choices of a^* : $0.1, 0.5, 1.0$. Based on progressive type II censored data we have obtained the point predictors and 95% PLs for the missing order statistics $Y_{k:r_j}, k = 1, ..., r_j, j = 1, ..., m$ in the one-sample prediction problem and for the kth order statistics $Y_{k:m_2:n_2}, k = 1, ..., m_2$ of unobserved progressively type II censoring sample in the two-sample prediction problem. Also, based on record data we have obtained the point predictors and 95% PLs for the future nth record $X_{U(n)}$ in the one-sample prediction problem and for the kth record $Y_{U(k)}$ of unobserved record sample of size n in the two-sample prediction problem.

5.1 Results Based on Progressively Type II Censored Data

In this section we present the results of the Bayes estimators for the Weibull parameters α and λ, and their corresponding credible interval lengths, when Prior 0 and 1 are used. As well as we present the results of one-sample and two-sample prediction problems. These results are reported in Tables 5.1-5.7, for the following schemes:

Scheme 1: $n = 30, m = 10$, $(r_1 = ... = r_4 = 5, r_5 = ... = r_{10} = 0)$, $(30, 10, 4^*5, 6^*0)$
Scheme 2: $n = 30, m = 15$, $(r_1 = ... = r_6 = 0, r_7 = r_8 = r_9 = 5, r_{10} = ... = r_{15} = 0)$, $(30, 15, 6^*0, 3^*5, 6^*0)$
Scheme 3: $n = 30, m = 20$, $(r_1 = ... = r_{18} = 0, r_{19} = r_{20} = 5)$, $(30, 20, 18^*0, 2^*5)$
Scheme 4: $n = 30, m = 25$, $(r_1 = ... = r_{24} = 0, r_{25} = 5)$, $(30, 25, 24^*0, 5)$
Scheme 5: $n = 40, m = 10$, $(r_1 = ... = r_{10} = 3)$, $(40, 10, 10^*3)$
Scheme 6: $n = 40, m = 20$, $(r_1 = ... = r_{10} = 2, r_{11} = ... = r_{20} = 0)$, $(40, 20, 10^*2, 10^*0)$.

In Table 5.1, we present the MLEs for α and λ as well as the Bayes estimates of α and λ, under the different loss functions L_1, L_2 and L_3 with different choices of a^* : $0.1, 1.0, 5.0$, when Prior 0 is used. The numerical results of the MLEs for α and λ were computed by using the equations (3.4) and (3.6). The numerical results of the Bayes estimators for α and λ, and their corresponding MSEs, were computed using the equations given in steps 4-6 of Algorithm 1. The codes of Mathematica 7 are used for this purpose which appear in Appendix [2].

In Table 5.2, we present the Bayes estimates of α and λ, under different loss functions L_1, L_2 and L_3 with different choices of $a^* : 0.1, 1.0, 5.0$, when Prior 1 is used.

In Table 5.3, we show numerical comparisons between the average lengths of the credible intervals of α and λ when Prior 0 and 1 are used for all schemes considered. The average length of the credible intervals for α and λ were computed by using step 7 of Algorithm 1.

In Table 5.4, we present the point predictors and PIs for the missing kth order statistics $Y_{k:r_j}, k = 1, ..., r_j, j = 1, ..., m$, based on observed progressive type II censoring sample of size m with censoring scheme $(r_1, r_2, ..., r_m)$, for all schemes described above and for all different loss functions L_1, L_2 and L_3 with different choices of $a^* : 0.1, 0.5, 1.0$. Based on MCMC samples $\{(\alpha_i, \lambda_i), i = 1, 2, ..., M\}$ obtained by using steps 1-3 of Algorithm 1 and $M = 1000$, the point predictors for the missing order statistics $Y_{k:r_j}$ in censoring stage j, $k = 1, 2, ..., r_j$, were computed under different loss functions, by using the equations (3.20), (3.23) and (3.24), respectively. The 95% lower bound L and upper bound U of PI for the missing kth order statistics $Y_{k:r_j}$ were computed by solving the equations (3.27) and (3.28) with respect to L and U, respectively. The codes of Mathematica 7 are used for this purpose, see Appendix [3].

In Table 5.5, we present the performances of one-sample Bayes predictors when Prior 0 and 1 are used, for scheme $(15, 10, 5^*0, 5, 4^*0)$.

In Table 5.6, we present the point predictors and PIs for the future kth order statistics $Y_{k:m_2:n_2}, k = 1, ..., m_2$ with censoring scheme $(s_1, s_2, ..., s_{m_2})$, based on observed progressive type II censoring sample $\{x_{i:m_1:n_1}, i = 1, ..., m_1\}$ with censoring scheme $(r_1, r_2, ..., r_{m_1})$, and for different loss functions L_1, L_2 and L_3 with different choices of $a^* : 0.1, 0.5, 1.0$. Based on MCMC samples $\{(\alpha_i, \lambda_i), i = 1, 2, ..., M\}$ and $M = 1000$, the point predictors for $Y_{k:m_2:n_2}$, $k = 1, 2, ..., m_2$, were computed under different loss functions, by using the equations (3.32), (3.35) and (3.36), respectively. The 95% lower bound L and upper bound U of PI for the future kth order statistics $Y_{k:m_2:n_2}$ were computed by solving the equations (3.38) and (3.39) with respect to L and U, respectively. The codes of Mathematica 7 are used for this purpose, see Appendix [4].

In Table 5.7, we present the performances of Bayes predictors based on two-sample prediction problem when Prior 0 and 1 are used, for observed progressive scheme $(30, 10, 4^*5, 6^*0)$, and unobserved progressive scheme $(10, 6, 4, 5^*0)$.

In Tables 5.4 and 5.6, the smallest, middle or around, and the largest missing kth order statistics $Y_{k:r_j}$ (future kth order statistics $Y_{k:m_2:n_2}$) are only predicted.

Table 5.1: MLEs and Bayes estimates with respect to different loss functions when Prior 0 is used, for $\alpha = 2$ and $\lambda = 1$. (progressive data)

Censoring schemes		MLE	Sq. err. Bayes 1	Abs. err. Bayes 2	$a^* = 0.1$ Bayes 3	$a^* = 1$ Bayes 4	$a^* = 5$ Bayes 5
	α	2.2691	2.1169	2.1283	2.1163	2.1157	2.1131
Scheme 1		(0.2614)	(0.2377)	(0.2388)	(0.2377)	(0.2378)	(0.2381)
	λ	1.2180	1.1778	1.1392	1.1152	1.0633	0.8416
$(30, 10, 4^*5, 6^*0)$		(0.3358)	(0.2651)	(0.2457)	(0.2400)	(0.2200)	(0.1450)
	α	2.1858	2.0794	2.0858	2.0792	2.0790	2.0782
Scheme 2		(0.1926)	(0.1788)	(0.1789)	(0.1788)	(0.1788)	(0.1789)
	λ	1.1491	1.1257	1.1025	1.0851	1.0516	0.9029
$(30, 15, 6^*0, 3^*5, 6^*0)$		(0.1782)	(0.1519)	(0.1459)	(0.1411)	(0.1324)	(0.0983)
	α	2.1665	2.0585	2.0627	2.0584	2.0584	2.0580
Scheme 3		(0.1589)	(0.1458)	(0.1461)	(0.1458)	(0.1458)	(0.1458)
	λ	1.0785	1.0584	1.0413	1.0301	1.0068	0.9033
$(30, 20, 18^*0, 2^*5)$		(0.0657)	(0.0561)	(0.0545)	(0.0532)	(0.0508)	(0.0410)
	α	2.1128	2.0334	2.0334	2.0333	2.0333	2.0332
Scheme 4		(0.1469)	(0.1371)	(0.1371)	(0.1371)	(0.1371)	(0.1371)
	λ	1.0458	1.0129	1.0129	0.9998	0.9895	0.9550
$(30, 25, 24^*0, 5)$		(0.0495)	(0.0422)	(0.0401)	(0.0399)	(0.0378)	(0.0361)
	α	2.3112	2.1118	2.1211	2.1114	2.111	2.1091
Scheme 5		(0.3484)	(0.3026)	(0.3028)	(0.3027)	(0.3028)	(0.3032)
	λ	1.3746	1.2064	1.1616	1.1390	1.0826	0.8380
$(40, 10, 10^*3)$		(0.8557)	(0.5114)	(0.4591)	(0.4485)	(0.3980)	(0.2127)
	α	2.2001	2.1251	2.1297	2.1250	2.1249	2.1246
Scheme 6		(0.1598)	(0.1505)	(0.1508)	(0.1505)	(0.1505)	(0.1506)
	λ	1.0514	1.0487	1.0318	1.0204	0.9970	0.8929
$(40, 20, 10^*2, 10^*0)$		(0.0690)	(0.0665)	(0.0644)	(0.0629)	(0.0560)	(0.0488)

Note: The first entry represents the average estimate and the second entry is the MSE.

Table 5.2: Bayes estimates with respect to different loss functions when Prior 1 is used, for $\alpha = 2$ and $\lambda = 1$. (progressive data)

Censoring schemes		Sq. err. Bayes 1	Abs. err. Bayes 2	$a^* = 0.1$ Bayes 3	$a^* = 1$ Bayes 4	$a^* = 5$ Bayes 5
Scheme 1	α	2.0811	2.0912	2.0806	2.0802	2.0782
		(0.1836)	(0.1834)	(0.1837)	(0.1837)	(0.1840)
	λ	1.1293	1.0996	1.0747	1.0292	0.8339
$(30, 10, 4^*5, 6^*0)$		(0.1728)	(0.1631)	(0.1568)	(0.1444)	(0.1011)
Scheme 2	α	2.0716	2.0716	2.0715	2.0715	2.0711
		(0.1712)	(0.1711)	(0.1712)	(0.1712)	(0.1713)
	λ	1.0822	1.0595	1.0459	1.0159	0.8816
$(30, 15, 6^*0, 3^*5, 6^*0)$		(0.1094)	(0.1094)	(0.1053)	(0.1030)	(0.0988)
Scheme 3	α	2.0509	2.0601	2.0511	2.0533	2.0500
		(0.1222)	(0.1223)	(0.1222)	(0.1222)	(0.1222)
	λ	1.0178	1.0019	0.9911	0.9692	0.8722
$(30, 20, 18^*0, 2^*5)$		(0.0391)	(0.0379)	(0.0368)	(0.0351)	(0.0287)
Scheme 4	α	2.0306	2.0335	2.0305	2.0305	2.0303
		(0.1180)	(0.1181)	(0.1180)	(0.1180)	(0.1180)
	λ	1.0485	1.0341	1.0269	1.0091	0.9289
$(30, 25, 24^*0, 5)$		(0.0323)	(0.0325)	(0.0310)	(0.0299)	(0.0251)
Scheme 5	α	2.0598	2.0676	2.0595	2.0592	2.0578
		(0.2026)	(0.2026)	(0.2026)	(0.2027)	(0.2029)
	λ	1.1886	1.1500	1.12920	1.0800	0.8693
$(40, 10, 10^*3)$		(0.1881)	(0.1736)	(0.1692)	(0.1543)	(0.0999)
Scheme 6	α	2.0424	2.0466	2.0423	2.0422	2.0419
		(0.1275)	(0.1274)	(0.1275)	(0.1275)	(0.1275)
	λ	1.0866	1.0696	1.0584	1.0351	0.9318
$(40, 20, 10^*2, 10^*0)$		(0.0563)	(0.0558)	(0.0537)	(0.0516)	(0.0441)

Table 5.3: Average credible interval lengths (A.L) and coverage percentages (C.P). (progressive data)

Schemes		Prior 0		Prior 1	
		A.L	C.P	A.L	C.P
Scheme 1	α	1.8146	0.94	1.5742	0.96
$(30, 10, 4^*5, 6^*0)$	λ	2.1901	0.93	1.7882	0.93
Scheme 2	α	1.6437	0.93	1.4718	0.94
$(30, 15, 6^*0, 3^*5, 6^*0)$	λ	1.6808	0.96	1.5166	0.94
Scheme 3	α	1.5450	0.94	1.4227	0.94
$(30, 20, 18^*0, 2^*5)$	λ	1.3090	0.94	1.2033	0.94
Scheme 4	α	1.3954	0.94	1.2806	0.94
$(30, 25, 24^*0, 5)$	λ	1.1741	0.94	1.1022	0.95
Scheme 5	α	2.3486	0.94	1.6960	0.93
$(40, 10, 10^*3)$	λ	3.6185	0.94	1.8811	0.96
Scheme 6	α	1.3199	0.96	1.3561	0.93
$(40, 20, 10^*2, 10^*0)$	λ	1.3678	0.94	1.2693	0.95

Comments and Observations

From the previous tables, we may observe the following remarks:

(1) From Table 5.1 and the simulated values for the MLEs, it is clear that as m increases the performances of MLEs of α and λ become better in terms of biases and MSEs.

(2) From Table 5.1, we observe that the Bayes estimates of α and λ obtained by using Prior 0 and with respect to different loss functions L_1, L_2 and L_3, are quite close to each other and perform much better than the MLEs of α and λ in terms of biases and MSEs for all schemes considered.

(3) It is observed that from Table 5.1 most of the different Bayes estimates 1-5 usually overestimate α and λ, except Bayes estimate 5 which underestimate α and λ. One can also observe that for each Bayes estimate, the biases and MSEs decrease as the progressive sample size m increases and for fixed the sample size n in the most schemes considered. But for fixed m as n increases the performance of Bayes estimates 1-5 becomes worse in terms of biases and MSEs in the most of schemes considered.

(4) If we compare the results in Table 5.2 by the results in Table 5.1, it observed that the Bayes estimates of α and λ obtained by using Prior 1 (informative prior) perform much better than the Bayes estimates of α and λ obtained by using Prior 0 (non-informative prior) in terms of the biases and MSEs in the most of schemes considered. Also we notice that the Bayes estimates obtained by using Prior 1 perform much better than the MLEs of α and λ in all schemes considered.

(5) From Tables 5.3, we observe that the average length of the credible intervals for α and λ, when Prior 1 is used, becomes smaller as expected, and decreases as m increases. For both Prior 0 and 1, the simulated probabilities for 0.95 are quite close to 0.95.

Table 5.4: Point predictors and PIs for missing kth order statistics $Y_{k:r_j}, k = 1, ..., r_j, j = 1, ..., m$, based on one-sample prediction problem.

Schemes	$Y_{k:r_j}$	Loss function	Predicted values	95%PIs
$(30, 10, 4^*5, 6^*0)$	$Y_{1:r_1}$	Sq. err.	0.4363	$(0.2150, 0.8323)$
		Ab. err.	0.4059	$(0.2139, 0.7673)$
		LINEX ($a^* = 0.1$)	0.4048	$(0.2150, 0.7744)$
		LINEX ($a^* = 0.5$)	0.3941	$(0.2150, 0.7550)$
		LINEX ($a^* = 1.0$)	0.3816	$(0.2150, 0.7328)$
	$Y_{3:r_1}$	Sq. err.	0.7925	$(0.4136, 1.3110)$
		Ab. err.	0.7686	$(0.4103, 1.2390)$
		LINEX ($a^* = 0.1$)	0.7565	$(0.4131, 1.1670)$
		LINEX ($a^* = 0.5$)	0.7435	$(0.4130, 1.1260)$
		LINEX ($a^* = 1.0$)	0.7272	$(0.4128, 1.0790)$
	$Y_{5:r_1}$	Sq. err.	1.2740	$(0.7145, 2.0560)$
		Ab. err.	1.2350	$(0.7571, 1.8980)$
		LINEX ($a^* = 0.1$)	1.2240	$(0.7131, 1.8340)$
		LINEX ($a^* = 0.5$)	1.2070	$(0.7126, 1.7650)$
		LINEX ($a^* = 1.0$)	1.1850	$(0.7120, 1.6890)$
	$Y_{1:r_2}$	Sq. err.	0.4722	$(0.2841, 0.8448)$
		Ab. err.	0.4393	$(0.2858, 0.9871)$
		LINEX ($a^* = 0.1$)	0.4487	$(0.2841, 0.7847)$
		LINEX ($a^* = 0.5$)	0.4410	$(0.2841, 0.7657)$
		LINEX ($a^* = 1.0$)	0.4319	$(0.2841, 0.7444)$
	$Y_{3:r_2}$	Sq. err.	0.8075	$(0.4454, 1.3170)$
		Ab. err.	0.7820	$(0.4253, 1.0920)$
		LINEX ($a^* = 0.1$)	0.7744	$(0.4449, 1.1650)$
		LINEX ($a^* = 0.5$)	0.7626	$(0.4447, 1.1150)$
		LINEX ($a^* = 1.0$)	0.7479	$(0.4445, 1.0620)$
	$Y_{5:r_2}$	Sq. err.	1.2830	$(0.7322, 2.0560)$
		Ab. err.	1.2440	$(0.7831, 1.9340)$
		LINEX ($a^* = 0.1$)	1.2350	$(0.7308, 1.8370)$
		LINEX ($a^* = 0.5$)	1.2180	$(0.7303, 1.7650)$
		LINEX ($a^* = 1.0$)	1.1970	$(0.7296, 1.6860)$

Schemes	$Y_{k:r_j}$	Loss function	Predicted values	95%PIs
	$Y_{1:r_3}$	Sq. err.	0.4887	$(0.3125, 0.8496)$
		Ab. err.	0.4555	$(0.3128, 0.8271)$
		LINEX $(a^* = 0.1)$	0.4679	$(0.3124, 0.7898)$
		LINEX $(a^* = 0.5)$	0.4611	$(0.3124, 0.7706)$
		LINEX $(a^* = 1.0)$	0.4531	$(0.3124, 0.7482)$
	$Y_{3:r_3}$	Sq. err.	0.8274	$(0.4677, 1.3360)$
		Ab. err.	0.8017	$(0.4345, 1.0430)$
		LINEX $(a^* = 0.1)$	0.7954	$(0.4672, 1.1900)$
		LINEX $(a^* = 0.5)$	0.7840	$(0.4670, 1.1350)$
		LINEX $(a^* = 1.0)$	0.7699	$(0.4668, 1.0580)$
	$Y_{5:r_3}$	Sq. err.	1.2750	$(0.7355, 2.0340)$
		Ab. err.	1.2360	$(0.7312, 1.7710)$
		LINEX $(a^* = 0.1)$	1.2290	$(0.7340, 1.8230)$
		LINEX $(a^* = 0.5)$	1.2120	$(0.7334, 1.7480)$
		LINEX $(a^* = 1.0)$	1.1920	$(0.7327, 1.6470)$
	$Y_{1:r_4}$	Sq. err.	0.5591	$(0.4105, 0.8934)$
		Ab. err.	0.5254	$(0.4104, 0.8316)$
		LINEX $(a^* = 0.1)$	0.5445	$(0.4105, 0.8321)$
		LINEX $(a^* = 0.5)$	0.5397	$(0.4105, 0.8127)$
		LINEX $(a^* = 1.0)$	0.5341	$(0.4105, 0.7907)$
	$Y_{3:r_4}$	Sq. err.	0.8497	$(0.5285, 1.3280)$
		Ab. err.	0.8225	$(0.4933, 1.0100)$
		LINEX $(a^* = 0.1)$	0.8236	$(0.5281, 1.1890)$
		LINEX $(a^* = 0.5)$	0.8144	$(0.5280, 1.1480)$
		LINEX $(a^* = 1.0)$	0.8033	$(0.5278, 1.1040)$
	$Y_{5:r_4}$	Sq. err.	1.3050	$(0.7769, 2.0690)$
		Ab. err.	1.2640	$(0.8537, 2.1080)$
		LINEX $(a^* = 0.1)$	1.2610	$(0.7756, 1.8440)$
		LINEX $(a^* = 0.5)$	1.2450	$(0.7752, 1.7760)$
		LINEX $(a^* = 1.0)$	1.2260	$(0.7746, 1.7040)$

Schemes	$Y_{k:r_j}$	Loss function	Predicted values	95%PIs
$(30, 15, 6^*0, 3^*5, 6^*0)$	$Y_{1:r_7}$	Sq. err.	0.9321	$(0.7721, 1.2975)$
		Ab. err.	0.8943	$(0.7729, 1.3395)$
		LINEX $(a^* = 0.1)$	0.9213	$(0.7721, 1.2490)$
		LINEX $(a^* = 0.5)$	0.9177	$(0.7721, 1.2320)$
		LINEX $(a^* = 1.0)$	0.9134	$(0.7721, 1.2112)$
	$Y_{3:r_7}$	Sq. err.	1.2661	$(0.9025, 1.7865)$
		Ab. err.	1.2388	$(0.9106, 1.7292)$
		LINEX $(a^* = 0.1)$	1.2443	$(0.9022, 1.6780)$
		LINEX $(a^* = 0.5)$	1.2366	$(0.9021, 1.6399)$
		LINEX $(a^* = 1.0)$	1.2272	$(0.9020, 1.5880)$
	$Y_{5:r_7}$	Sq. err.	1.7729	$(1.1870, 2.5739)$
		Ab. err.	1.7353	$(1.1260, 2.3966)$
		LINEX $(a^* = 0.1)$	1.7349	$(1.1860, 2.4036)$
		LINEX $(a^* = 0.5)$	1.7214	$(1.1850, 2.3469)$
		LINEX $(a^* = 1.0)$	1.7047	$(1.1850, 2.2809)$
	$Y_{1:r_8}$	Sq. err.	0.9563	$(0.8028, 1.3122)$
		Ab. err.	0.9191	$(0.8027, 1.2613)$
		LINEX $(a^* = 0.1)$	0.9464	$(0.8028, 1.2645)$
		LINEX $(a^* = 0.5)$	0.9431	$(0.8028, 1.2486)$
		LINEX $(a^* = 1.0)$	0.9391	$(0.8028, 1.2298)$
	$Y_{3:r_8}$	Sq. err.	1.2842	$(0.9270, 1.8051)$
		Ab. err.	1.2557	$(0.9094, 1.6219)$
		LINEX $(a^* = 0.1)$	1.2631	$(0.9268, 1.6886)$
		LINEX $(a^* = 0.5)$	1.2556	$(0.9267, 1.6506)$
		LINEX $(a^* = 1.0)$	1.2465	$(0.9265, 1.6075)$
	$Y_{5:r_8}$	Sq. err.	1.7954	$(1.2100, 2.5958)$
		Ab. err.	1.7577	$(1.2560, 2.5369)$
		LINEX $(a^* = 0.1)$	1.7580	$(1.2090, 2.4310)$
		LINEX $(a^* = 0.5)$	1.7447	$(1.2090, 2.3740)$
		LINEX $(a^* = 1.0)$	1.7283	$(1.2080, 2.2965)$

Schemes	$Y_{k:r_j}$	Loss function	Predicted values	95%PIs
	$Y_{1:r_9}$	Sq. err.	0.9845	$(0.8382, 1.3282)$
		Ab. err.	0.9482	$(0.8387, 1.3375)$
		LINEX $(a^* = 0.1)$	0.9756	$(0.8382, 1.2821)$
		LINEX $(a^* = 0.5)$	0.9726	$(0.8382, 1.2658)$
		LINEX $(a^* = 1.0)$	0.9690	$(0.8382, 1.2456)$
	$Y_{3:r_9}$	Sq. err.	1.3019	$(0.9567, 1.8017)$
		Ab. err.	1.2748	$(0.9846, 1.9012)$
		LINEX $(a^* = 0.1)$	1.2825	$(0.9565, 1.7041)$
		LINEX $(a^* = 0.5)$	1.2756	$(0.9564, 1.6677)$
		LINEX $(a^* = 1.0)$	1.2673	$(0.9563, 1.6229)$
	$Y_{5:r_9}$	Sq. err.	1.7952	$(1.2220, 2.5947)$
		Ab. err.	1.7556	$(1.1440, 2.2186)$
		LINEX $(a^* = 0.1)$	1.7586	$(1.2210, 2.4160)$
		LINEX $(a^* = 0.5)$	1.7456	$(1.2200, 2.3600)$
		LINEX $(a^* = 1.0)$	1.7297	$(1.2200, 2.2961)$
$(30, 20, 18^*0, 2^*5)$	$Y_{1:r_{19}}$	Sq. err.	1.1079	$(1.0030, 1.3862)$
		Ab. err.	1.0766	$(1.0028, 1.3364)$
		LINEX $(a^* = 0.1)$	1.1030	$(1.0030, 1.3543)$
		LINEX $(a^* = 0.5)$	1.1014	$(1.0030, 1.3440)$
		LINEX $(a^* = 1.0)$	1.0994	$(1.0030, 1.3322)$
	$Y_{3:r_{19}}$	Sq. err.	1.3780	$(1.0836, 1.8705)$
		Ab. err.	1.3430	$(1.0803, 1.7529)$
		LINEX $(a^* = 0.1)$	1.3623	$(1.0835, 1.7878)$
		LINEX $(a^* = 0.5)$	1.3569	$(1.0835, 1.7600)$
		LINEX $(a^* = 1.0)$	1.3503	$(1.0834, 1.7279)$
	$Y_{5:r_{19}}$	Sq. err.	1.9163	$(1.3061, 2.8734)$
		Ab. err.	1.8553	$(1.3626, 2.9124)$
		LINEX $(a^* = 0.1)$	1.8724	$(1.3053, 2.6926)$
		LINEX $(a^* = 0.5)$	1.8572	$(1.3050, 2.6273)$
		LINEX $(a^* = 1.0)$	1.8387	$(1.3046, 2.5498)$

Schemes	$Y_{k:r_j}$	Loss function	Predicted values	95%PIs
	$Y_{1:r_{20}}$	Sq. err.	1.2521	$(1.1580, 1.5061)$
		Ab. err.	1.2233	$(1.1588, 1.5812)$
		LINEX $(a^* = 0.1)$	1.2485	$(1.1580, 1.4769)$
		LINEX $(a^* = 0.5)$	1.2472	$(1.1580, 1.4669)$
		LINEX $(a^* = 1.0)$	1.2457	$(1.1580, 1.4551)$
	$Y_{3:r_{20}}$	Sq. err.	1.4992	$(1.2291, 1.9647)$
		Ab. err.	1.4648	$(1.2423, 1.9810)$
		LINEX $(a^* = 0.1)$	1.4866	$(1.2289, 1.8818)$
		LINEX $(a^* = 0.5)$	1.4822	$(1.2289, 1.8527)$
		LINEX $(a^* = 1.0)$	1.4768	$(1.2288, 1.8167)$
	$Y_{5:r_{20}}$	Sq. err.	1.9996	$(1.4287, 2.9159)$
		Ab. err.	1.9389	$(1.4380, 2.7437)$
		LINEX $(a^* = 0.1)$	1.9618	$(1.4281, 2.7453)$
		LINEX $(a^* = 0.5)$	1.9487	$(1.4278, 2.6864)$
		LINEX $(a^* = 1.0)$	1.9328	$(1.4275, 2.6157)$
$(30, 25, 24^*0, 5)$	$Y_{1:r_{25}}$	Sq. err.	1.0625	$(1.0035, 1.2228)$
		Ab. err.	1.0443	$(1.0034, 1.1982)$
		LINEX $(a^* = 0.1)$	1.0608	$(1.0035, 1.2073)$
		LINEX $(a^* = 0.5)$	1.0602	$(1.0035, 1.2022)$
		LINEX $(a^* = 1.0)$	1.0594	$(1.0035, 1.1961)$
	$Y_{3:r_{25}}$	Sq. err.	1.2200	$(1.0481, 1.5180)$
		Ab. err.	1.1977	$(1.0504, 1.5661)$
		LINEX $(a^* = 0.1)$	1.2136	$(1.0480, 1.4737)$
		LINEX $(a^* = 0.5)$	1.2113	$(1.0480, 1.4585)$
		LINEX $(a^* = 1.0)$	1.2086	$(1.0480, 1.4408)$
	$Y_{5:r_{25}}$	Sq. err.	1.5562	$(1.1791, 2.1660)$
		Ab. err.	1.5153	$(1.1943, 2.2201)$
		LINEX $(a^* = 0.1)$	1.5347	$(1.1788, 2.0703)$
		LINEX $(a^* = 0.5)$	1.5272	$(1.1787, 2.0393)$
		LINEX $(a^* = 1.0)$	1.5182	$(1.1785, 2.0031)$

Schemes	$Y_{k:r_j}$	Loss function	Predicted values	95%PIs
$(40, 10, 10^*3)$	$Y_{1:r_1}$	Sq. err.	0.5961	$(0.2995, 1.0293)$
		Ab. err.	0.5739	$(0.3006, 0.9930)$
		LINEX $(a^* = 0.1)$	0.5622	$(0.2995, 0.9711)$
		LINEX $(a^* = 0.5)$	0.5501	$(0.2995, 0.9523)$
		LINEX $(a^* = 1.0)$	0.5355	$(0.2994, 0.9316)$
	$Y_{3:r_1}$	Sq. err.	1.1020	$(0.6388, 1.6773)$
		Ab. err.	1.0820	$(0.5868, 1.5104)$
		LINEX $(a^* = 0.1)$	1.0670	$(0.6382, 1.5494)$
		LINEX $(a^* = 0.5)$	1.0540	$(0.6380, 1.5072)$
		LINEX $(a^* = 1.0)$	1.0370	$(0.6377, 1.4588)$
	$Y_{1:r_5}$	Sq. err.	0.7103	$(0.5214, 1.0773)$
		Ab. err.	0.6782	$(0.5215, 1.0380)$
		LINEX $(a^* = 0.1)$	0.6943	$(0.5214, 1.0189)$
		LINEX $(a^* = 0.5)$	0.6890	$(0.5214, 1.0002)$
		LINEX $(a^* = 1.0)$	0.6825	$(0.5213, 0.9783)$
	$Y_{3:r_5}$	Sq. err.	1.1400	$(0.7243, 1.6870)$
		Ab. err.	1.1170	$(0.7262, 1.6236)$
		LINEX $(a^* = 0.1)$	1.1110	$(0.7238, 1.5661)$
		LINEX $(a^* = 0.5)$	1.1000	$(0.7236, 1.5218)$
		LINEX $(a^* = 1.0)$	1.0880	$(0.7233, 1.4646)$
	$Y_{1:r_{10}}$	Sq. err.	0.8944	$(0.7769, 1.1735)$
		Ab. err.	0.8649	$(0.7772, 1.2003)$
		LINEX $(a^* = 0.1)$	0.8880	$(0.7769, 1.1217)$
		LINEX $(a^* = 0.5)$	0.8858	$(0.7769, 1.1046)$
		LINEX $(a^* = 1.0)$	0.8831	$(0.7769, 1.0843)$
	$Y_{3:r_{10}}$	Sq. err.	1.2340	$(0.8946, 1.7409)$
		Ab. err.	1.2050	$(0.9035, 1.6428)$
		LINEX $(a^* = 0.1)$	1.2140	$(0.8942, 1.6170)$
		LINEX $(a^* = 0.5)$	1.2070	$(0.8941, 1.5770)$
		LINEX $(a^* = 1.0)$	1.1980	$(0.8939, 1.5313)$

Schemes	$Y_{k:r_j}$	Loss function	Predicted values	95%PIs
$(40, 20, 10^*2, 10^*0)$	$Y_{1:r_1}$	Sq. err.	0.6663	$(0.2116, 1.4672)$
		Ab. err.	0.6062	$(0.2150, 1.5160)$
		LINEX ($a^* = 0.1$)	0.5791	$(0.2116, 1.3949)$
		LINEX ($a^* = 0.5$)	0.5490	$(0.2116, 1.3711)$
		LINEX ($a^* = 1.0$)	0.5135	$(0.2116, 1.3424)$
	$Y_{2:r_1}$	Sq. err.	1.1970	$(0.4315, 2.3160)$
		Ab. err.	1.1350	$(0.4346, 2.3537)$
		LINEX ($a^* = 0.1$)	1.0880	$(0.4313, 2.1809)$
		LINEX ($a^* = 0.5$)	1.0470	$(0.4312, 2.1334)$
		LINEX ($a^* = 1.0$)	0.9964	$(0.4311, 2.0745)$
	$Y_{1:r_5}$	Sq. err.	0.7196	$(0.3206, 1.4823)$
		Ab. err.	0.6551	$(0.3201, 1.4058)$
		LINEX ($a^* = 0.1$)	0.6531	$(0.3206, 1.4130)$
		LINEX ($a^* = 0.5$)	0.6311	$(0.3206, 1.3883)$
		LINEX ($a^* = 1.0$)	0.6055	$(0.3206, 1.3585)$
	$Y_{2:r_5}$	Sq. err.	1.2360	$(0.5001, 2.3489)$
		Ab. err.	1.1710	$(0.4962, 2.1681)$
		LINEX ($a^* = 0.1$)	1.1370	$(0.4999, 2.2070)$
		LINEX ($a^* = 0.5$)	1.1010	$(0.4998, 2.1597)$
		LINEX ($a^* = 1.0$)	1.0570	$(0.4997, 2.1046)$
	$Y_{1:r_{10}}$	Sq. err.	0.9344	$(0.6336, 1.6072)$
		Ab. err.	0.8665	$(0.6323, 1.4567)$
		LINEX ($a^* = 0.1$)	0.8994	$(0.6335, 1.5394)$
		LINEX ($a^* = 0.5$)	0.8881	$(0.6335, 1.5159)$
		LINEX ($a^* = 1.0$)	0.8748	$(0.6335, 1.4872)$
	$Y_{2:r_{10}}$	Sq. err.	1.3750	$(0.7469, 2.4133)$
		Ab. err.	1.3040	$(0.7585, 2.4078)$
		LINEX ($a^* = 0.1$)	1.3050	$(0.7467, 2.2797)$
		LINEX ($a^* = 0.5$)	1.2810	$(0.7467, 2.2331)$
		LINEX ($a^* = 1.0$)	1.2530	$(0.7466, 2.1781)$

Schemes	$Y_{k:r_j}$	Loss function	Predicted values	95%PIs
$(40, 30, 14^{*}0, 10, 15^{*}0)$	$Y_{1:r_{15}}$	Sq. err.	0.6705	$(0.6103, 0.8245)$
		Ab. err.	0.6535	$(0.6102, 0.8054)$
		LINEX $(a^{*} = 0.1)$	0.6680	$(0.6103, 0.8125)$
		LINEX $(a^{*} = 0.5)$	0.6671	$(0.6103, 0.8084)$
		LINEX $(a^{*} = 1.0)$	0.6661	$(0.6103, 0.8036)$
	$Y_{2:r_{15}}$	Sq. err.	0.7336	$(0.6253, 0.9312)$
		Ab. err.	0.7176	$(0.6249, 0.9044)$
		LINEX $(a^{*} = 0.1)$	0.7290	$(0.6253, 0.9108)$
		LINEX $(a^{*} = 0.5)$	0.7274	$(0.6253, 0.9038)$
		LINEX $(a^{*} = 1.0)$	0.7254	$(0.6253, 0.8957)$
	$Y_{5:r_{15}}$	Sq. err.	0.9334	$(0.7285, 1.2166)$
		Ab. err.	0.9196	$(0.7422, 1.2134)$
		LINEX $(a^{*} = 0.1)$	0.9243	$(0.7283, 1.1707)$
		LINEX $(a^{*} = 0.5)$	0.9211	$(0.7282, 1.1540)$
		LINEX $(a^{*} = 1.0)$	0.9172	$(0.7281, 1.1345)$
	$Y_{6:r_{15}}$	Sq. err.	1.0130	$(0.7780, 1.3305)$
		Ab. err.	0.9992	$(0.7837, 1.2983)$
		LINEX $(a^{*} = 0.1)$	1.0030	$(0.7777, 1.2728)$
		LINEX $(a^{*} = 0.5)$	0.9991	$(0.7776, 1.2539)$
		LINEX $(a^{*} = 1.0)$	0.9944	$(0.7775, 1.2329)$
	$Y_{9:r_{15}}$	Sq. err.	1.3440	$(0.9839, 1.8189)$
		Ab. err.	1.3240	$(1.0650, 1.9212)$
		LINEX $(a^{*} = 0.1)$	1.3260	$(0.9832, 1.7292)$
		LINEX $(a^{*} = 0.5)$	1.3190	$(0.9829, 1.6991)$
		LINEX $(a^{*} = 1.0)$	1.3110	$(0.9826, 1.6644)$
	$Y_{10:r_{15}}$	Sq. err.	1.5740	$(1.1030, 2.2323)$
		Ab. err.	1.5410	$(1.2060, 2.4523)$
		LINEX $(a^{*} = 0.1)$	1.5460	$(1.1020, 2.1185)$
		LINEX $(a^{*} = 0.5)$	1.5360	$(1.1020, 2.0823)$
		LINEX $(a^{*} = 1.0)$	1.5240	$(1.1020, 2.0414)$

Table 5.5: Average values and MSEs of the predictors of missing values based on one-sample prediction problem for scheme $(15, 10, 5^*0, 5, 4^*0)$.

$Y_{k:r_j}$	Loss function	Prior 0		Prior 1	
		Average predicted values	MSE	Average predicted values	MSE
$Y_{1:r_6}$	Sq. err.	0.8354	0.0206	0.7681	0.0171
	Ab. err.	0.7973	0.0205	0.7347	0.0176
	LINEX ($a^* = 0.1$)	0.8254	0.0209	0.7593	0.0176
	LINEX ($a^* = 0.5$)	0.8221	0.0210	0.7564	0.0178
	LINEX ($a^* = 1.0$)	0.8183	0.0211	0.7531	0.0180
$Y_{2:r_6}$	Sq. err.	1.0120	0.0256	0.9547	0.0175
	Ab. err.	0.9600	0.0230	0.9154	0.0161
	LINEX ($a^* = 0.1$)	0.9922	0.0254	0.9360	0.0169
	LINEX ($a^* = 0.5$)	0.9856	0.0253	0.9298	0.0168
	LINEX ($a^* = 1.0$)	0.9778	0.0253	0.9223	0.0167
$Y_{3:r_6}$	Sq. err.	1.1190	0.0413	1.1000	0.0298
	Ab. err.	1.0790	0.0362	1.0570	0.0271
	LINEX ($a^* = 0.1$)	1.0930	0.0375	1.0710	0.0286
	LINEX ($a^* = 0.5$)	1.0850	0.0363	1.0620	0.0284
	LINEX ($a^* = 1.0$)	1.0740	0.0351	1.0500	0.0281
$Y_{4:r_6}$	Sq. err.	1.3630	0.0722	1.3530	0.0544
	Ab. err.	1.3050	0.0566	1.3000	0.0467
	LINEX ($a^* = 0.1$)	1.3180	0.0589	1.3120	0.0490
	LINEX ($a^* = 0.5$)	1.3030	0.0551	1.2980	0.0474
	LINEX ($a^* = 1.0$)	1.2840	0.0509	1.2820	0.0457
$Y_{5:r_6}$	Sq. err.	1.6470	0.1725	1.7100	0.1144
	Ab. err.	1.5720	0.1349	1.6320	0.0946
	LINEX ($a^* = 0.1$)	1.5830	0.1381	1.6410	0.0968
	LINEX ($a^* = 0.5$)	1.5620	0.1280	1.6180	0.0916
	LINEX ($a^* = 1.0$)	1.5360	0.1167	1.5900	0.0858

Table 5.6: Point predictors and PIs for future kth order statistics $Y_{k:m_2:n_2}, k = 1, ..., m_2$, based on another independent observed sample $\{x_{i:m_1:n_1}, i = 1, ..., m_1\}$.

Schemes of observed sample	Schemes of future sample	$Y_{k:m_2:n_2}$	Loss function	Predicted values	95%PIs
$(30, 10, 4^*5, 6^*0)$	$(10, 6, 4, 5^*0)$	$Y_{1:6:10}$	Sq. err.	0.2768	$(0.0575, 0.5917)$
			Ab. err.	0.2605	$(0.0658, 0.6253)$
			LINEX $(a^* = 0.1)$	0.2341	$(0.0591, 0.5937)$
			LINEX $(a^* = 0.5)$	0.2157	$(0.0597, 0.5945)$
			LINEX $(a^* = 1.0)$	0.1889	$(0.0605, 0.5955)$
		$Y_{3:6:10}$	Sq. err.	0.6446	$(0.3038, 1.1016)$
			Ab. err.	0.6244	$(0.3519, 1.1374)$
			LINEX $(a^* = 0.1)$	0.6089	$(0.3224, 1.1087)$
			LINEX $(a^* = 0.5)$	0.5956	$(0.3304, 1.1115)$
			LINEX $(a^* = 1.0)$	0.5788	$(0.3419, 1.1152)$
		$Y_{6:6:10}$	Sq. err.	1.2680	$(0.7110, 2.0527)$
			Ab. err.	1.2282	$(0.8323, 2.0978)$
			LINEX $(a^* = 0.1)$	1.2186	$(0.7753, 2.0689)$
			LINEX $(a^* = 0.5)$	1.2008	$(0.8074, 2.0755)$
			LINEX $(a^* = 1.0)$	1.1789	$(0.8604, 2.0846)$
$(30, 15, 6^*0, 3^*5, 6^*0)$	$(20, 8, 3^*0, 2^*4, 3^*0)$	$Y_{1:8:20}$	Sq. err.	0.2914	$(0.0767, 0.5625)$
			Ab. err.	0.2819	$(0.0758, 0.5339)$
			LINEX $(a^* = 0.1)$	0.2572	$(0.0778, 0.5634)$
			LINEX $(a^* = 0.5)$	0.2425	$(0.0782, 0.5638)$
			LINEX $(a^* = 1.0)$	0.2213	$(0.0787, 0.5642)$
		$Y_{4:8:20}$	Sq. err.	0.5650	$(0.3365, 0.8345)$
			Ab. err.	0.5579	$(0.3366, 0.7775)$
			LINEX $(a^* = 0.1)$	0.5491	$(0.3510, 0.8373)$
			LINEX $(a^* = 0.5)$	0.5431	$(0.3570, 0.8384)$
			LINEX $(a^* = 1.0)$	0.5356	$(0.3649, 0.8397)$
		$Y_{8:8:20}$	Sq. err.	1.3800	$(0.8407, 2.0930)$
			Ab. err.	1.3500	$(0.8415, 1.9550)$
			LINEX $(a^* = 0.1)$	1.3400	$(0.8857, 2.0990)$
			LINEX $(a^* = 0.5)$	1.3250	$(0.9054, 2.1010)$
			LINEX $(a^* = 1.0)$	1.3070	$(0.9319, 2.1040)$

Schemes of observed sample	Schemes of future sample	$Y_{k:m_2:n_2}$	Loss function	Predicted values	95%PIs
$(30, 20, 18^{*}0, 2^{*}5)$	$(20, 10, 8^{*}0, 2^{*}5)$	$Y_{1:10:20}$	Sq. err.	0.1944	$(0.0366, 0.4221)$
			Ab. err.	0.1823	$(0.0397, 0.4408)$
			LINEX$(a^{*} = 0.1)$	0.1619	$(0.0371, 0.4227)$
			LINEX $(a^{*} = 0.5)$	0.1477	$(0.0373, 0.4229)$
			LINEX $(a^{*} = 1.0)$	0.1266	$(0.0375, 0.4232)$
		$Y_{5:10:20}$	Sq. err.	0.4911	$(0.2804, 0.7487)$
			Ab. err.	0.4829	$(0.3094, 0.7650)$
			LINEX $(a^{*} = 0.1)$	0.4749	$(0.2930, 0.7511)$
			LINEX $(a^{*} = 0.5)$	0.4689	$(0.2981, 0.7520)$
			LINEX $(a^{*} = 1.0)$	0.4612	$(0.3048, 0.7532)$
		$Y_{10:10:20}$	Sq. err.	0.7963	$(0.5247, 1.1380)$
			Ab. err.	0.7841	$(0.5852, 1.1530)$
			LINEX $(a^{*} = 0.1)$	0.7796	$(0.5601, 1.1430)$
			LINEX $(a^{*} = 0.5)$	0.7735	$(0.5755, 1.1450)$
			LINEX $(a^{*} = 1.0)$	0.7659	$(0.5968, 1.1470)$
$(30, 25, 24^{*}0, 5)$	$(20, 15, 14^{*}0, 5)$	$Y_{1:15:20}$	Sq. err.	0.2994	$(0.0890, 0.5457)$
			Ab. err.	0.2935	$(0.0956, 0.5727)$
			LINEX$(a^{*} = 0.1)$	0.2697	$(0.0897, 0.5462)$
			LINEX $(a^{*} = 0.5)$	0.2569	$(0.0899, 0.5463)$
			LINEX $(a^{*} = 1.0)$	0.2386	$(0.0902, 0.5465)$
		$Y_{8:15:20}$	Sq. err.	0.7547	$(0.5484, 0.9850)$
			Ab. err.	0.7506	$(0.5984, 1.0130)$
			LINEX $(a^{*} = 0.1)$	0.7456	$(0.5678, 0.9875)$
			LINEX $(a^{*} = 0.5)$	0.7423	$(0.5760, 0.9884)$
			LINEX $(a^{*} = 1.0)$	0.7381	$(0.5871, 0.9897)$
		$Y_{15:15:20}$	Sq. err.	1.0790	$(0.8387, 1.3540)$
			Ab. err.	1.0720	$(0.9221, 1.3830)$
			LINEX $(a^{*} = 0.1)$	1.0700	$(0.8804, 1.3590)$
			LINEX $(a^{*} = 0.5)$	1.0670	$(0.8996, 1.3610)$
			LINEX $(a^{*} = 1.0)$	1.0630	$(0.9263, 1.3640)$

Table 5.7: Average values and MSEs of the predictors of future values based on two-sample prediction problem for observed progressive scheme $(30, 10, 4^*5, 6^*0)$ and future progressive scheme $(10, 6, 4, 5^*0)$.

$Y_{k:m_2:n_2}$	Loss function	Prior 0		Prior 1	
		Average predicted values	MSE	Average predicted values	MSE
$Y_{1:6:10}$	Sq. err.	0.2950	0.0055	0.2898	0.0039
	Ab. err.	0.2677	0.0090	0.2686	0.0044
	LINEX ($a^* = 0.1$)	0.2443	0.0062	0.2396	0.0041
	LINEX ($a^* = 0.5$)	0.2225	0.0066	0.2180	0.0043
	LINEX ($a^* = 1.0$)	0.1901	0.0074	0.1862	0.0047
$Y_{2:6:10}$	Sq. err.	0.5272	0.0089	0.5241	0.0079
	Ab. err.	0.4979	0.0089	0.4962	0.0083
	LINEX ($a^* = 0.1$)	0.4751	0.0086	0.4729	0.0081
	LINEX ($a^* = 0.5$)	0.4554	0.0086	0.4534	0.0083
	LINEX ($a^* = 1.0$)	0.4296	0.0088	0.4279	0.0087
$Y_{3:6:10}$	Sq. err.	0.7303	0.0148	0.7153	0.0119
	Ab. err.	0.6995	0.0149	0.6857	0.0115
	LINEX ($a^* = 0.1$)	0.6805	0.0146	0.6659	0.0112
	LINEX ($a^* = 0.5$)	0.6621	0.0147	0.6476	0.0110
	LINEX ($a^* = 1.0$)	0.6387	0.0149	0.6243	0.0109
$Y_{4:6:10}$	Sq. err.	0.9187	0.0275	0.9313	0.0200
	Ab. err.	0.8796	0.0254	0.8947	0.0182
	LINEX ($a^* = 0.1$)	0.8610	0.0244	0.8765	0.0176
	LINEX ($a^* = 0.5$)	0.8400	0.0236	0.8565	0.0169
	LINEX ($a^* = 1.0$)	0.8137	0.0227	0.8314	0.0163
$Y_{5:6:10}$	Sq. err.	1.1850	0.0437	1.1390	0.0293
	Ab. err.	1.1390	0.0366	1.0960	0.0253
	LINEX ($a^* = 0.1$)	1.1240	0.0350	1.0810	0.0243
	LINEX ($a^* = 0.5$)	1.1020	0.0325	1.0600	0.0229
	LINEX ($a^* = 1.0$)	1.0750	0.0298	1.0340	0.0214
$Y_{6:6:10}$	Sq. err.	1.6050	0.1081	1.5480	0.0722
	Ab. err.	1.5290	0.0895	1.4800	0.0575
	LINEX ($a^* = 0.1$)	1.5140	0.0860	1.4640	0.0552
	LINEX ($a^* = 0.5$)	1.4830	0.0793	1.4350	0.0501
	LINEX ($a^* = 1.0$)	1.4430	0.0718	1.3990	0.0444

Comments and observations

From Tables 5.4-5.7, we may observe the following remarks:

(1) It is observed that from Table 5.4, the predicted values for the missing kth order statistics $Y_{k:r_j}$ based on different loss functions L_1, L_2 and L_3, and at any censoring stage, are quite close to each other and fall in their corresponding 95% PIs for all schemes considered in the table.

(2) It is observed that from Tables 5.5 and 5.7, the performances of the Bayes predictors for the missing (future) order statistics when Prior 1 is used, are better than the performances of the Bayes predictors when Prior 0 is used.

(3) From Table 5.6, we notice that the behavior of the predicted values for the future kth order statistics $Y_{k:m_2:n_2}$ based on different loss functions and for all schemes considered in the table, are similar to the predicted values in Table 5.4.

Example 1: (real data)

In this example we analyze the time (in minutes) to breakdown of an insulating fluid between electrodes at voltage 30kv. This data is taken from Nelson (1982, Table 6.1, p.228). The complete data set consist of $n = 11$ times to breakdown. The progressively censored data we used is as follows:

i	1	2	3	4	5	6	7	8
r_i	0	0	0	0	3	0	0	0
y_i	2.0464	2.8361	3.0184	3.0454	3.1206	4.9706	5.1698	5.2724

Before we analyze the data, we have subtracted 1.75 from each data point. After subtracting 1.75 from each data point, we have computed the maximum likelihood estimators of α and λ and they are 2.0239 and 0.1802, respectively. The corresponding Kolmogorov-Smirnov (KS) distance becomes 0.1656 and the associated p-value is 0.9149. Therefore the KS indicates that the Weibull distribution can be used to analyze this data.

Now we compute the Bayes estimates with respect to different loss function described in subsection [1.3.1], namely squared error (Sq. err.), absolute error (Ab. err.) and LINEX function with different choices of $a^* : 0.1, 1.0, 5.0$. The results are presented in Table 5.8. All the estimates are quite close to each other.

We obtain the 95% credible intervals of α and λ and they are $(1.5571, 1.7324)$ and $(0.0986, 0.4246)$, respectively. It is observed that all Bayes estimates of α and λ are falling in their corresponding credible intervals.
Also, we consider the prediction of the 1^{st}, 2^{nd} and 3^{rd} order statistics in stage 5, which are missing. The predicted values and the 95% PI of the 1^{st}, 2^{nd} and 3^{rd} order statistics are presented in Table 5.9. It is observed that all predicted values, with respect to different loss functions, are all ordered and fall in their corresponding predictive intervals.

Table 5.8: Bayes estimates with respect to different loss functions for the data in example 1.

	Sq. err. Bayes 1	Abs. err. Bayes 2	$a^* = 0.1$ Bayes 3	$a^* = 1.0$ Bayes 4	$a^* = 5$ Bayes 5
α	1.6779	1.6897	1.6771	1.6764	1.673
λ	0.23286	0.22354	0.21677	0.20328	0.14342

Table 5.9: Point predictors and 95% PIs for the missing data in example 1 .

Scheme	$Y_{k:r_j}$	Loss function	Predicted values	95%PIs
$(11, 8, 4^*0, 3, 3^*0)$	$Y_{1:r_5}$	Sq. err.	3.782	$(3.143, 5.189)$
		Ab. err.	3.644	$(3.159, 5.498)$
		LINEX $(a^* = 0.1)$	3.741	$(3.143, 4.801)$
		LINEX $(a^* = 0.5)$	3.728	$(3.143, 4.709)$
		LINEX $(a^* = 1.0)$	3.711	$(3.143, 4.622)$
	$Y_{2:r_5}$	Sq. err.	4.401	$(3.343, 6.043)$
		Ab. err.	4.300	$(3.396, 5.948)$
		LINEX $(a^* = 0.1)$	4.342	$(3.342, 5.532)$
		LINEX $(a^* = 0.5)$	4.321	$(3.342, 5.391)$
		LINEX $(a^* = 1.0)$	4.296	$(3.342, 5.258)$
	$Y_{3:r_5}$	Sq. err.	5.201	$(3.751, 7.245)$
		Ab. err.	5.099	$(3.874, 7.362)$
		LINEX $(a^* = 0.1)$	5.120	$(3.749, 6.589)$
		LINEX $(a^* = 0.5)$	5.091	$(3.748, 6.414)$
		LINEX $(a^* = 1.0)$	5.055	$(3.747, 6.240)$

We can estimate the density function of α and λ by fitting the curve on the histogram of MCMC samples $\{\alpha_i, i = 1, ..., M\}$ and $\{\lambda_i, i = 1, ..., M\}$ generated by Algorithm 1, respectively. Based on $M = 10000$ replications of generation for α and λ, the estimate density functions of α and λ are shown in the following figures:

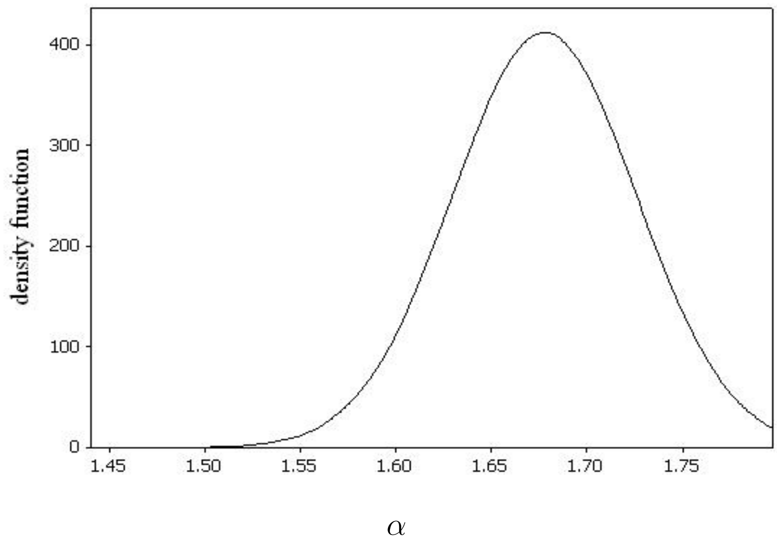

Figure 5.1: Estimate of the density function for α, based on the progressive data in example 1.

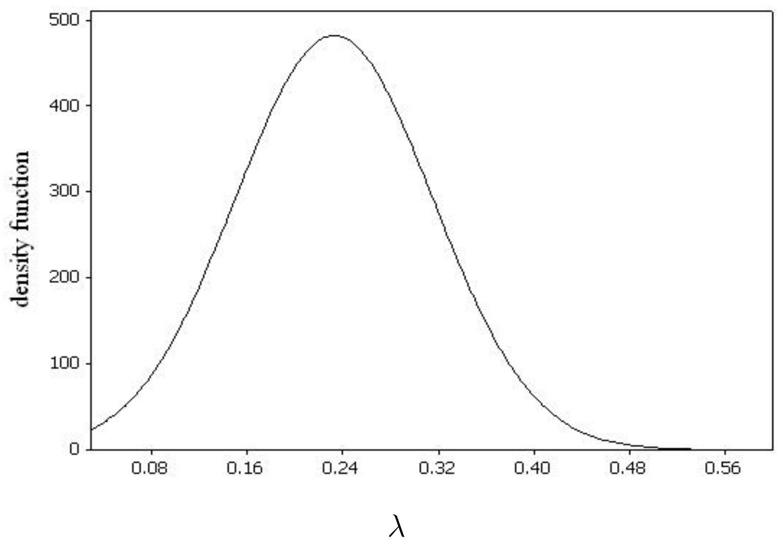

Figure 5.2: Estimate of the density function for λ, based on the progressive data in example 1.

5.2 Results Based on Record Data

In this section we present the results of MLEs and Bayes estimators for α and λ when the shape parameter α is unknown as well as the results of prediction in one-sample and two-sample prediction problems. These results are reported in Tables 5.10-5.16, for the following cases of sample sizes:

Case 1: $n = 6$
Case 2: $n = 9$
Case 3: $n = 12$
Case 4: $n = 15$

In Table 5.10, we present the MLEs of α and λ as well as the Bayes estimates of α and λ, under the different loss functions used in previous section, when Prior 0 is used. The MLE of the shape parameter α of Weibull distribution was computed by solving Eq.(4.4). The MLE of the scale parameter λ of Weibull distribution was computed by using Eq.(4.5). Based on different loss functions L_1, L_2 and L_3, the Bayes estimates and their corresponding MSEs for the shape parameter α and the scale parameter λ of Weibull distribution, were computed by using the equations given in steps 4-6 of Algorithm 1. The MSEs were computed based on $M = 1000$ replications. The codes of Mathematica 7 are used for this purpose which appear in Appendix [5].

In Table 5.11, we present the Bayes estimates of α and λ, under the different loss functions, when Prior 1 is used.

In Table 5.12, we show numerical comparisons between the average lengths of the credible intervals of α and λ when Prior 0 and 1 are used for all cases considered.

In Table 5.13, we present the point predictors and PIs for the future nth record $X_{U(n)}, 1 \leq m < n$, based on observed record sample of size m, for all cases described above and for all different loss functions L_1, L_2 and L_3 with different choices of $a^* : 0.1, 0.5, 1.0$. Based on MCMC samples $\{(\alpha_i, \lambda_i), i = 1, 2, ..., M\}$ and $M = 1000$, the predicted values for the future nth record $X_{U(n)}$, $1 \leq m < n$, were computed under different loss functions, by using the equations (4.15), (4.18) and (4.19), respectively.

The 95% lower bound L and upper bound U of the prediction interval for the future nth record were computed by solving the equations (4.24) and (4.25) with respect to L and U, respectively. In this table the first five future nth records after the last observed record are only predicted. The codes of Mathematica 7 are used for this purpose, see Appendix [6].

In Table 5.14, we present the performances of one-sample Bayes predictors when Prior 0 and 1 are used for some cases of m.

In Table 5.15, we present the point predictors and PIs for the unobserved kth record $Y_{U(k)}, k = 1, ..., n$, based on observed record sample of size m, and for all different loss functions used in previous Tables. Based on MCMC samples $\{(\alpha_i, \lambda_i), i = 1, 2, ..., M\}$ and $M = 1000$, the predicted values for the unobserved kth record $Y_{U(k)}$, were computed under different loss functions, by using the equations (4.28), (4.33) and (4.34), respectively. The 95% lower bound L and upper bound U of the prediction interval for the unobserved kth record $Y_{U(k)}$ were computed by solving the equations (4.36) and (4.37) with respect to L and U, respectively. In this table the smallest, middle or around, and the largest unobserved kth record $Y_{U(k)}$ are only predicted. The codes of Mathematica 7 are used for this purpose, see Appendix [7].

In Table 5.16, we present the performances of two-sample Bayes predictors when Prior 0 and 1 are used for some cases of m and n.

Table 5.10: MLEs and Bayes estimates with respect to different loss functions when Prior 0 is used, for $\alpha = 2$ and $\lambda = 1$. (record data)

Cases		MLE	Sq. err. Bayes 1	Abs. err. Bayes 2	$a^* = 0.1$ Bayes 3	$a^* = 1.0$ Bayes 4	$a^* = 5$ Bayes 5
$n = 6$	α	3.0691	2.2599	2.3358	2.1801	2.0603	1.5709
		(3.0490)	(0.2801)	(0.3107)	(0.3171)	(0.4648)	(1.7100)
	λ	0.9000	1.4838	1.3380	1.2965	1.1515	0.7055
		(0.6953)	(0.8299)	(0.8628)	(0.8794)	(0.9968)	(1.7600)
$n = 9$	α	2.7464	2.2033	2.2668	2.1662	2.1247	1.9549
		(1.3480)	(0.1386)	(0.1579)	(0.1542)	(0.2582)	(0.4830)
	λ	0.9038	1.4023	1.2971	1.2764	1.1894	0.8755
		(0.5460)	(0.5533)	(0.5909)	(0.5901)	(0.6433)	(1.0180)
$n = 12$	α	2.4559	2.1342	2.1609	2.1280	2.1214	2.0912
		(0.6759)	(0.0244)	(0.0282)	(0.0258)	(0.0307)	(0.0701)
	λ	0.9038	1.2390	1.1943	1.1698	1.1167	0.8981
		(0.5321)	(0.2441)	(0.2515)	(0.2535)	(0.2697)	(0.4353)
$n = 15$	α	2.3462	2.1120	2.1282	2.1100	2.1074	2.0798
		(0.5217)	(0.0075)	(0.0077)	(0.0074)	(0.0080)	(0.0700)
	λ	0.9132	1.1936	1.1618	1.1464	1.1083	0.9455
		(0.4167)	(0.1473)	(0.1498)	(0.1504)	(0.1573)	(0.2325)

Note: The first entry represents the average estimate and the second entry is the MSE.

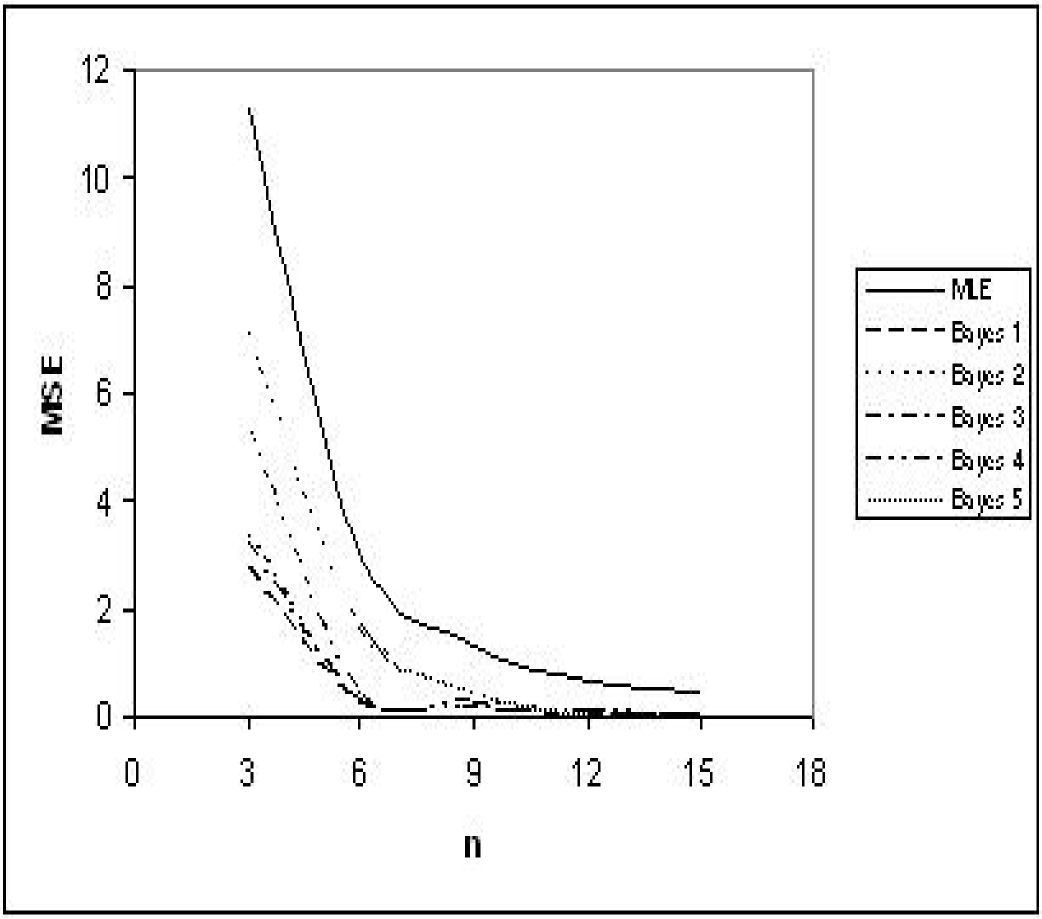

Figure 5.3: MSEs for different estimators of α, in Table 5.10, for cases 1-5.

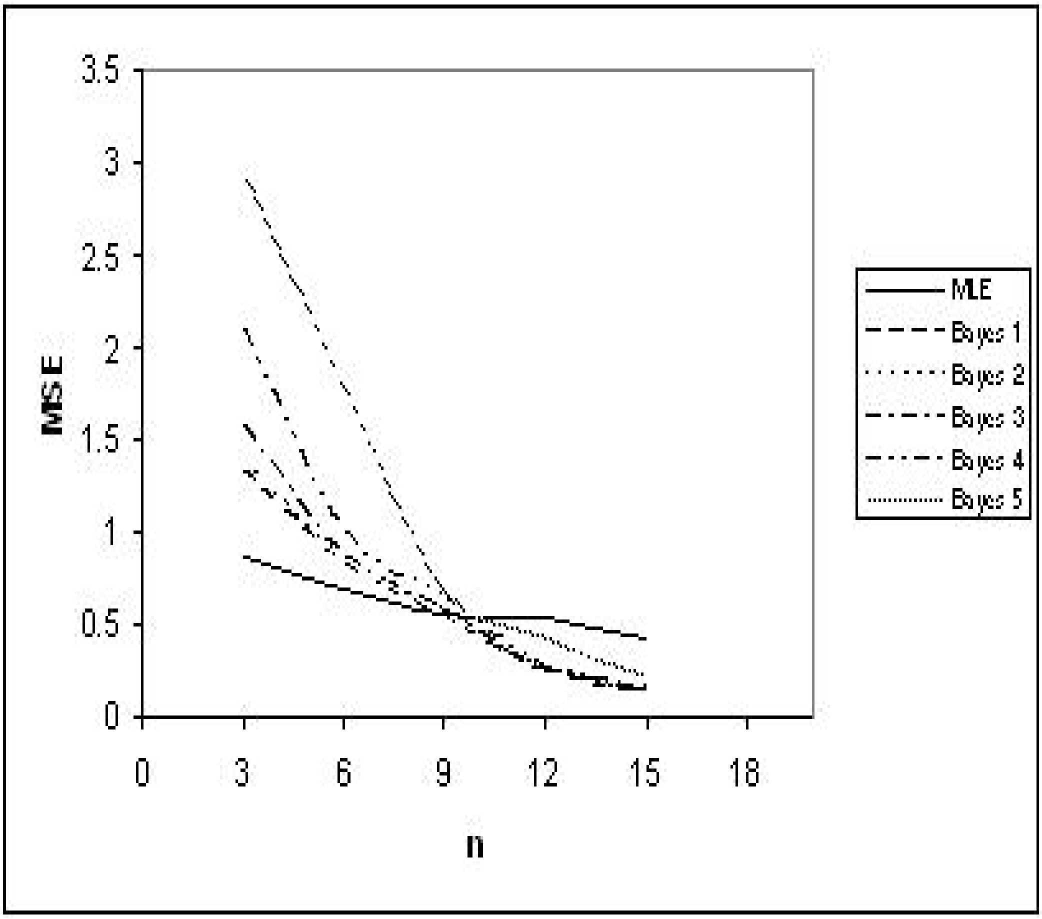

Figure 5.4: MSEs for different estimators of λ, in Table 5.10, for cases 1-5.

Table 5.11: Bayes estimates with respect to different loss functions when Prior 1 is used, for $\alpha = 2$ and $\lambda = 1$. (record data)

Cases		Sq. err. Bayes 1	Abs. err. Bayes 2	$a^* = 0.1$ Bayes 3	$a^* = 1.0$ Bayes 4	$a^* = 5$ Bayes 5
$n = 6$	α	1.9145	1.9498	1.9070	1.8999	1.8498
		(0.0236)	(0.0251)	(0.0236)	(0.0239)	(0.0331)
	λ	1.2254	1.1583	1.1277	1.0473	0.7332
		(0.2606)	(0.2673)	(0.2716)	(0.2970)	(0.5380)
$n = 9$	α	1.9946	2.0169	1.9918	1.9894	1.9765
		(0.0097)	(0.0103)	(0.0097)	(0.0097)	(0.0101)
	λ	1.2294	1.1804	1.1578	1.0990	0.8569
		(0.1815)	(0.1849)	(0.1873)	(0.2009)	(0.3469)
$n = 12$	α	1.9981	2.0138	1.9969	1.9958	1.9906
		(0.0043)	(0.0046)	(0.0043)	(0.0043)	(0.0044)
	λ	1.1163	1.0815	1.0670	1.0265	0.8535
		(0.1151)	(0.1169)	(0.1178)	(0.1244)	(0.1968)
$n = 15$	α	1.9753	1.9870	1.9746	1.9740	1.9712
		(0.0025)	(0.0027)	(0.0025)	(0.0025)	(0.0025)
	λ	1.1724	1.1455	1.1308	1.0965	0.9443
		(0.1036)	(0.1052)	(0.1056)	(0.1103)	(0.1642)

Table 5.12: Average credible interval lengths (A.L) and coverage percentages (C.P). (record data)

Cases		Prior 0		Prior 1	
		A.L	C.P	A.L	C.P
$n = 6$	α	4.8235	0.94	1.8495	0.94
	λ	4.8064	0.92	2.5439	0.95
$n = 9$	α	3.0842	0.94	1.5988	0.96
	λ	3.8603	0.96	2.2788	0.96
$n = 12$	α	2.5697	0.94	1.4280	0.93
	λ	3.4542	0.94	2.2107	0.94
$n = 15$	α	2.2221	0.95	1.3082	0.96
	λ	3.2821	0.94	2.1062	0.96

Comments and observations

From Tables 5.10-5.12, we may observe the following remarks:

(1) From Table 5.10, we observe that as n increases the performances of MLEs of α and λ become better in terms of biases and MSEs.

(2) From Figures 5.3-5.4, it is observed that all Bayes estimates of the shape parameter α using Prior 0 are quite close to each other. These estimates compete the corresponding MLEs in the sense of biases and MSEs. For the scale parameter, its Bayes estimator of λ works similarly for $n > 10$, approximately.

(3) It can be noticed that the estimates overestimate the parameters α and λ except for the MLE and Bayes estimate of λ, which underestimate λ in all cases considered. One can also notice that for each Bayes estimates 1-5, the biases and MSEs decrease as the record sample size n increases in the most cases considered.

(4) The Bayes estimates of α and λ obtained by using Prior 1 (informative prior) perform well comparing the corresponding ones obtained by using Prior 0 (non-informative prior). Also, these Bayes estimates perform well comparing the MLEs in all cases considered.

(5) From Table 5.12, it is evident that the average length of the credible intervals for α and λ, when Prior 1 is used, becomes smaller as expected, and decreases as n increases. For both Prior 0 and 1, the simulated probabilities for 0.95 are quite close to 0.95.

Table 5.13: Point predictors and PIs for future records $X_{U(n)}, 1 \leq m < n$ based on some observed records.

Size of observed sample	$X_{U(n)}$	Loss function	Predicted values	95%PIs
$m = 6$	$X_{U(7)}$	Sq. err.	2.1333	(1.9673, 2.6302)
		Ab. err.	2.0756	(1.9687, 2.6223)
		LINEX ($a^* = 0.1$)	2.1256	(1.9673, 2.4558)
		LINEX ($a^* = 0.5$)	2.1231	(1.9673, 2.4086)
		LINEX ($a^* = 1.0$)	2.1200	(1.9673, 2.3601)
	$X_{U(8)}$	Sq. err.	2.2893	(1.9328, 2.9626)
		Ab. err.	2.2234	(2.0146, 2.8916)
		LINEX ($a^* = 0.1$)	2.2751	(1.9329, 2.6543)
		LINEX ($a^* = 0.5$)	2.2703	(1.9330, 2.5738)
		LINEX ($a^* = 1.0$)	2.2646	(1.9331, 2.4969)
	$X_{U(9)}$	Sq. err.	2.4341	(2.0500, 3.2423)
		Ab. err.	2.3617	(2.0913, 3.1052)
		LINEX ($a^* = 0.1$)	2.4141	(2.0495, 2.8104)
		LINEX ($a^* = 0.5$)	2.4074	(2.0493, 2.6998)
		LINEX ($a^* = 1.0$)	2.3993	(2.0491, 2.6010)
	$X_{U(10)}$	Sq. err.	2.5697	(2.1087, 3.4931)
		Ab. err.	2.4912	(2.1826, 3.2896)
		LINEX ($a^* = 0.1$)	2.5445	(2.1076, 2.9440)
		LINEX ($a^* = 0.5$)	2.5361	(2.1072, 2.8055)
		LINEX ($a^* = 1.0$)	2.5259	(2.1067, 2.6888)
	$X_{U(11)}$	Sq. err.	2.6976	(2.1705, 3.7241)
		Ab. err.	2.6132	(2.2806, 3.4548)
		LINEX ($a^* = 0.1$)	2.6677	(2.1685, 3.0629)
		LINEX ($a^* = 0.5$)	2.6577	(2.1678, 2.8978)
		LINEX ($a^* = 1.0$)	2.6456	(2.1669, 2.7665)

Size of observed sample	$X_{U(n)}$	Loss function	Predicted values	95%PIs
$m = 9$	$X_{U(10)}$	Sq. err.	3.5548	$(3.3724, 4.0820)$
		Ab. err.	3.4939	$(3.3733, 4.0956)$
		LINEX $(a^* = 0.1)$	3.5494	$(3.3724, 3.9631)$
		LINEX $(a^* = 0.5)$	3.5475	$(3.3724, 3.9267)$
		LINEX $(a^* = 1.0)$	3.5452	$(3.3724, 3.8875)$
	$X_{U(11)}$	Sq. err.	3.7323	$(3.4096, 4.4388)$
		Ab. err.	3.6651	$(3.4191, 4.4243)$
		LINEX $(a^* = 0.1)$	3.7220	$(3.4095, 4.2242)$
		LINEX $(a^* = 0.5)$	3.7184	$(3.4095, 4.1604)$
		LINEX $(a^* = 1.0)$	3.7140	$(3.4094, 4.0951)$
	$X_{U(12)}$	Sq. err.	3.9016	$(3.4707, 4.7422)$
		Ab. err.	3.8302	$(3.4967, 4.6948)$
		LINEX $(a^* = 0.1)$	3.8868	$(3.4703, 4.4369)$
		LINEX $(a^* = 0.5)$	3.8817	$(3.4702, 4.3474)$
		LINEX $(a^* = 1.0)$	3.8755	$(3.4700, 4.2600)$
	$X_{U(13)}$	Sq. err.	4.0637	$(3.5432, 5.0169)$
		Ab. err.	3.9885	$(3.5908, 4.9345)$
		LINEX $(a^* = 0.1)$	4.0449	$(3.5424, 4.6237)$
		LINEX $(a^* = 0.5)$	4.0384	$(3.5421, 4.5099)$
		LINEX $(a^* = 1.0)$	4.0305	$(3.5418, 4.4030)$
	$X_{U(14)}$	Sq. err.	4.2195	$(3.6214, 5.2720)$
		Ab. err.	4.1407	$(3.6939, 5.1536)$
		LINEX $(a^* = 0.1)$	4.1970	$(3.6200, 4.7934)$
		LINEX $(a^* = 0.5)$	4.1892	$(3.6195, 4.6559)$
		LINEX $(a^* = 1.0)$	4.1797	$(3.6189, 4.5318)$

Size of observed sample	$X_{U(n)}$	Loss function	Predicted values	95%PIs
$m = 12$	$X_{U(13)}$	Sq. err.	4.1090	$(3.9249, 4.6445)$
		Ab. err.	4.0471	$(3.9251, 4.6751)$
		LINEX $(a^* = 0.1)$	4.1041	$(3.9249, 4.5364)$
		LINEX $(a^* = 0.5)$	4.1024	$(3.9249, 4.5059)$
		LINEX $(a^* = 1.0)$	4.1003	$(3.9249, 4.4732)$
	$X_{U(14)}$	Sq. err.	4.2908	$(3.9626, 5.0172)$
		Ab. err.	4.2215	$(3.9641, 5.0538)$
		LINEX $(a^* = 0.1)$	4.2813	$(3.9625, 4.8172)$
		LINEX $(a^* = 0.5)$	4.2780	$(3.9625, 4.7632)$
		LINEX $(a^* = 1.0)$	4.2740	$(3.9624, 4.7074)$
	$X_{U(15)}$	Sq. err.	4.4664	$(4.0253, 5.3395)$
		Ab. err.	4.3916	$(4.0288, 5.3742)$
		LINEX $(a^* = 0.1)$	4.4526	$(4.0250, 5.0500)$
		LINEX $(a^* = 0.5)$	4.4478	$(4.0249, 4.9738)$
		LINEX $(a^* = 1.0)$	4.4420	$(4.0247, 4.8972)$
	$X_{U(16)}$	Sq. err.	4.6365	$(4.1006, 5.6349)$
		Ab. err.	4.5567	$(4.1063, 5.6626)$
		LINEX $(a^* = 0.1)$	4.6187	$(4.1000, 5.2573)$
		LINEX $(a^* = 0.5)$	4.6125	$(4.0998, 5.1597)$
		LINEX $(a^* = 1.0)$	4.6050	$(4.0995, 5.0640)$
	$X_{U(17)}$	Sq. err.	4.8017	$(4.1827, 5.9121)$
		Ab. err.	4.7169	$(4.1907, 5.9290)$
		LINEX $(a^* = 0.1)$	4.7800	$(4.1817, 5.4476)$
		LINEX $(a^* = 0.5)$	4.7724	$(4.1814, 5.3291)$
		LINEX $(a^* = 1.0)$	4.7633	$(4.1809, 5.2156)$

Size of observed sample	$X_{U(n)}$	Loss function	Predicted values	95%PIs
$m = 15$	$X_{U(16)}$	Sq. err.	4.6615	$(4.4859, 5.1655)$
		Ab. err.	4.6033	$(4.4861, 5.1198)$
		LINEX $(a^* = 0.1)$	4.6576	$(4.4859, 5.0835)$
		LINEX $(a^* = 0.5)$	4.6562	$(4.4859, 5.0557)$
		LINEX $(a^* = 1.0)$	4.6546	$(4.4859, 5.0232)$
	$X_{U(17)}$	Sq. err.	4.8364	$(4.5224, 5.5153)$
		Ab. err.	4.7724	$(4.5249, 5.4257)$
		LINEX $(a^* = 0.1)$	4.8289	$(4.5223, 5.3639)$
		LINEX $(a^* = 0.5)$	4.8262	$(4.5223, 5.3127)$
		LINEX $(a^* = 1.0)$	4.8230	$(4.5223, 5.2545)$
	$X_{U(18)}$	Sq. err.	5.0068	$(4.5834, 5.8179)$
		Ab. err.	4.9388	$(4.5915, 5.6848)$
		LINEX $(a^* = 0.1)$	4.9958	$(4.5831, 5.5985)$
		LINEX $(a^* = 0.5)$	4.9920	$(4.5830, 5.5243)$
		LINEX $(a^* = 1.0)$	4.9872	$(4.5829, 5.4421)$
	$X_{U(19)}$	Sq. err.	5.1729	$(4.6571, 6.0960)$
		Ab. err.	5.1016	$(4.6736, 5.9194)$
		LINEX $(a^* = 0.1)$	5.1587	$(4.6565, 5.8091)$
		LINEX $(a^* = 0.5)$	5.1537	$(4.6563, 5.7118)$
		LINEX $(a^* = 1.0)$	5.1476	$(4.6560, 5.6068)$
	$X_{U(20)}$	Sq. err.	5.3352	$(4.7378, 6.3576)$
		Ab. err.	5.2608	$(4.7650, 6.1377)$
		LINEX $(a^* = 0.1)$	5.3179	$(4.7368, 6.0036)$
		LINEX $(a^* = 0.5)$	5.3118	$(4.7364, 5.8833)$
		LINEX $(a^* = 1.0)$	5.3043	$(4.7359, 5.7566)$

Table 5.14: Average values and MSEs of the predictors of future records based on one-sample prediction problem.

m	$X_{U(n)}$	Loss function	Prior 0		Prior 1	
			Average predicted values	MSE	Average predicted values	MSE
6	$X_{U(7)}$	Sq. err.	2.8022	0.3551	2.7330	0.2569
		Ab. err.	2.7298	0.3440	2.6485	0.2305
		LINEX ($a^* = 0.1$)	2.8390	0.3644	2.7194	0.2507
		LINEX ($a^* = 0.5$)	2.8273	0.3622	2.7150	0.2488
		LINEX ($a^* = 1.0$)	2.8156	0.3601	2.7098	0.2467
	$X_{U(8)}$	Sq. err.	3.0101	0.4661	2.8960	0.3452
		Ab. err.	2.9224	0.4531	2.7977	0.3032
		LINEX ($a^* = 0.1$)	3.0742	0.4634	2.8699	0.3284
		LINEX ($a^* = 0.5$)	3.0354	0.4703	2.8616	0.3242
		LINEX ($a^* = 1.0$)	3.0094	0.4627	2.8519	0.3196
	$X_{U(9)}$	Sq. err.	3.1904	0.4625	3.0461	0.4011
		Ab. err.	3.1122	0.4223	2.9319	0.3730
		LINEX ($a^* = 0.1$)	3.2526	0.5556	3.0064	0.4017
		LINEX ($a^* = 0.5$)	3.2177	0.4952	2.9935	0.3940
		LINEX ($a^* = 1.0$)	3.1853	0.4501	2.9783	0.3853
9	$X_{U(10)}$	Sq. err.	3.1844	0.2396	3.1558	0.2281
		Ab. err.	3.1065	0.2278	3.0957	0.2196
		LINEX ($a^* = 0.1$)	3.1734	0.2377	3.1496	0.2272
		LINEX ($a^* = 0.5$)	3.1698	0.2371	3.1475	0.2269
		LINEX ($a^* = 1.0$)	3.1656	0.2364	3.1450	0.2265
	$X_{U(11)}$	Sq. err.	3.4382	0.3027	3.2306	0.2155
		Ab. err.	3.3586	0.2901	3.1632	0.2000
		LINEX ($a^* = 0.1$)	3.4230	0.3002	3.2183	0.2118
		LINEX ($a^* = 0.5$)	3.4178	0.2994	3.2140	0.2105
		LINEX ($a^* = 1.0$)	3.4117	0.2985	3.2090	0.2091
	$X_{U(12)}$	Sq. err.	3.5480	0.4022	3.4624	0.2580
		Ab. err.	3.4598	0.3718	3.3870	0.2449
		LINEX ($a^* = 0.1$)	3.5246	0.3908	3.4444	0.2548
		LINEX ($a^* = 0.5$)	3.5168	0.3873	3.4385	0.2537
		LINEX ($a^* = 1.0$)	3.5076	0.3834	3.4313	0.2525
12	$X_{U(13)}$	Sq. err.	3.4548	0.3870	3.4799	0.2577
		Ab. err.	3.4042	0.3728	3.4311	0.2498
		LINEX ($a^* = 0.1$)	3.4506	0.3854	3.4761	0.2571
		LINEX ($a^* = 0.5$)	3.4491	0.3848	3.4748	0.2569
		LINEX ($a^* = 1.0$)	3.4473	0.3842	3.4732	0.2566
	$X_{U(14)}$	Sq. err.	3.7024	0.2821	3.7908	0.2458
		Ab. err.	3.6443	0.2701	3.7336	0.2387
		LINEX ($a^* = 0.1$)	3.6939	0.2796	3.7830	0.2448
		LINEX ($a^* = 0.5$)	3.6910	0.2788	3.7803	0.2445
		LINEX ($a^* = 1.0$)	3.6874	0.2779	3.7770	0.2441
	$X_{U(15)}$	Sq. err.	3.8192	0.3173	3.8573	0.2326
		Ab. err.	3.7621	0.3078	3.7983	0.2220
		LINEX ($a^* = 0.1$)	3.8086	0.3151	3.8463	0.2300
		LINEX ($a^* = 0.5$)	3.8050	0.3143	3.8425	0.2291
		LINEX ($a^* = 1.0$)	3.8005	0.3135	3.8378	0.2281

Table 5.15: Point predictors and PIs for unobserved records $Y_{U(k)}, k = 1, ..., n$, based on another independent observed record sample of size m.

Size of observed sample	Size of future sample	$Y_{U(k)}$	Loss function	Predicted values	95%PIs
$m = 6$	$n = 4$	$Y_{U(1)}$	Sq. err.	0.6060	$(0.1041, 1.3800)$
			Ab. err.	0.5597	$(0.1291, 1.3504)$
			LINEX ($a^* = 0.1$)	0.4952	$(0.1174, 1.3876)$
			LINEX ($a^* = 0.5$)	0.4458	$(0.1212, 1.3905)$
			LINEX ($a^* = 1.0$)	0.4321	$(0.1257, 1.3941)$
		$Y_{U(3)}$	Sq. err.	1.1146	$(0.4873, 2.0060)$
			Ab. err.	1.0685	$(0.5747, 1.8600)$
			LINEX ($a^* = 0.1$)	1.0408	$(0.5457, 2.0318)$
			LINEX ($a^* = 0.5$)	1.0135	$(0.5696, 2.0421)$
			LINEX ($a^* = 1.0$)	0.9783	$(0.6023, 2.0556)$
		$Y_{U(4)}$	Sq. err.	1.2947	$(0.6364, 2.2425)$
			Ab. err.	1.2436	$(0.7480, 2.0402)$
			LINEX ($a^* = 0.1$)	1.2249	$(0.7183, 2.2823)$
			LINEX ($a^* = 0.5$)	1.1997	$(0.7556, 2.2990)$
			LINEX ($a^* = 1.0$)	1.1680	$(0.8080, 2.3217)$
$m = 9$	$n = 6$	$Y_{U(1)}$	Sq. err.	0.7759	$(0.1239, 1.8204)$
			Ab. err.	0.7080	$(0.1201, 1.8631)$
			LINEX ($a^* = 0.1$)	0.6236	$(0.1310, 1.8290)$
			LINEX ($a^* = 0.5$)	0.5577	$(0.1335, 1.8323)$
			LINEX ($a^* = 1.0$)	0.4590	$(0.1367, 1.8364)$
		$Y_{U(3)}$	Sq. err.	1.4774	$(0.6281, 2.6714)$
			Ab. err.	1.4181	$(0.5693, 2.7147)$
			LINEX ($a^* = 0.1$)	1.3758	$(0.6835, 2.6973)$
			LINEX ($a^* = 0.5$)	1.3379	$(0.7080, 2.7074)$
			LINEX ($a^* = 1.0$)	1.2896	$(0.7435, 2.7206)$
		$Y_{U(6)}$	Sq. err.	2.1556	$(1.1792, 3.5315)$
			Ab. err.	2.0881	$(1.0410, 3.5227)$
			LINEX ($a^* = 0.1$)	2.0669	$(1.3274, 3.5877)$
			LINEX ($a^* = 0.5$)	2.0350	$(1.4068, 3.6101)$
			LINEX ($a^* = 1.0$)	1.9952	$(1.5338, 3.6394)$

Size of observed sample	Size of future sample	$Y_{U(k)}$	Loss function	Predicted values	95%PIs
$m = 12$	$n = 8$	$Y_{U(1)}$	Sq. err.	1.3220	$(0.3699, 2.5011)$
			Ab. err.	1.2843	$(0.3667, 2.4614)$
			LINEX $(a^* = 0.1)$	1.1779	$(0.3778, 2.5063)$
			LINEX $(a^* = 0.5)$	1.1163	$(0.3807, 2.5082)$
			LINEX $(a^* = 1.0)$	1.0281	$(0.3843, 2.5107)$
		$Y_{U(4)}$	Sq. err.	2.4178	$(1.4631, 3.5477)$
			Ab. err.	2.3874	$(1.4289, 3.4699)$
			LINEX $(a^* = 0.1)$	2.3527	$(1.5406, 3.5646)$
			LINEX $(a^* = 0.5)$	2.3285	$(1.5727, 3.5710)$
			LINEX $(a^* = 1.0)$	2.2980	$(1.6163, 3.5792)$
		$Y_{U(8)}$	Sq. err.	3.1758	$(2.2053, 4.3516)$
			Ab. err.	3.1401	$(2.1486, 4.2155)$
			LINEX $(a^* = 0.1)$	3.1243	$(2.3830, 4.3861)$
			LINEX $(a^* = 0.5)$	3.1056	$(2.4661, 4.3995)$
			LINEX $(a^* = 1.0)$	3.0822	$(2.5822, 4.4169)$
$m = 15$	$n = 10$	$Y_{U(1)}$	Sq. err.	0.8506	$(0.1321, 1.9876)$
			Ab. err.	0.7776	$(0.1236, 1.9819)$
			LINEX $(a^* = 0.1)$	0.6814	$(0.1357, 1.9925)$
			LINEX $(a^* = 0.5)$	0.6076	$(0.1371, 1.9944)$
			LINEX $(a^* = 1.0)$	0.4966	$(0.1387, 1.9966)$
		$Y_{U(5)}$	Sq. err.	2.1937	$(1.1605, 3.5537)$
			Ab. err.	2.1370	$(1.0196, 3.5541)$
			LINEX $(a^* = 0.1)$	2.1003	$(1.2435, 3.5775)$
			LINEX $(a^* = 0.5)$	2.0660	$(1.2800, 3.5866)$
			LINEX $(a^* = 1.0)$	2.0227	$(1.3320, 3.5984)$
		$Y_{U(10)}$	Sq. err.	3.2077	$(2.0147, 4.7808)$
			Ab. err.	3.1425	$(1.7467, 4.7213)$
			LINEX $(a^* = 0.1)$	3.1243	$(2.2288, 4.8347)$
			LINEX $(a^* = 0.5)$	3.0942	$(2.3387, 4.8561)$
			LINEX $(a^* = 1.0)$	3.0567	$(2.5026, 4.8842)$

Size of observed sample	Size of future sample	$Y_{U(k)}$	Loss function	Predicted values	95%PIs
$m = 20$	$n = 15$	$Y_{U(1)}$	Sq. err.	1.1702	$(0.2365, 2.4789)$
			Ab. err.	1.1056	$(0.2448, 2.4780)$
			LINEX $(a^* = 0.1)$	0.9874	$(0.2399, 2.4823)$
			LINEX $(a^* = 0.5)$	0.9076	$(0.2411, 2.4836)$
			LINEX $(a^* = 1.0)$	0.7900	$(0.2426, 2.4851)$
		$Y_{U(8)}$	Sq. err.	3.4087	$(2.2290, 4.8328)$
			Ab. err.	3.3661	$(2.3323, 4.7163)$
			LINEX $(a^* = 0.1)$	3.3375	$(2.3620, 4.8545)$
			LINEX $(a^* = 0.5)$	3.3115	$(2.4196, 4.8628)$
			LINEX $(a^* = 1.0)$	3.2788	$(2.4979, 4.8734)$
		$Y_{U(15)}$	Sq. err.	4.5942	$(3.3091, 6.1587)$
			Ab. err.	4.5457	$(3.4887, 5.9444)$
			LINEX $(a^* = 0.1)$	4.5317	$(3.5895, 6.2026)$
			LINEX $(a^* = 0.5)$	4.5090	$(3.7253, 6.2198)$
			LINEX $(a^* = 1.0)$	4.4807	$(3.9120, 6.2424)$

Table 5.16: Average values and MSEs of the predictors of unobserved records based on two-sample prediction problem.

m	n	$Y_{U(k)}$	Loss function	Prior 0		Prior 1	
				Average predicted values	MSE	Average predicted values	MSE
6	3	$Y_{U(1)}$	Sq. err.	0.9682	0.1606	0.9001	0.0518
			Ab. err.	0.9085	0.1907	0.8215	0.0446
			LINEX ($a^* = 0.1$)	0.7859	0.1898	0.7299	0.0386
			LINEX ($a^* = 0.5$)	0.6891	0.2195	0.6571	0.0353
			LINEX ($a^* = 1.0$)	0.5463	0.2596	0.5478	0.0339
		$Y_{U(2)}$	Sq. err.	1.3559	0.1830	1.1970	0.0626
			Ab. err.	1.2322	0.3151	1.1268	0.0558
			LINEX ($a^* = 0.1$)	1.1547	0.3041	1.0673	0.0535
			LINEX ($a^* = 0.5$)	1.0866	0.3274	1.0172	0.0523
			LINEX ($a^* = 1.0$)	0.9855	0.3771	0.9512	0.0531
		$Y_{U(3)}$	Sq. err.	1.7434	0.2927	1.7264	0.0390
			Ab. err.	1.5613	0.4185	1.6352	0.0403
			LINEX ($a^* = 0.1$)	1.4703	0.4755	1.5860	0.0446
			LINEX ($a^* = 0.5$)	1.3566	0.5972	1.5346	0.0504
			LINEX ($a^* = 1.0$)	1.2526	0.6474	1.4692	0.0607
9	6	$Y_{U(1)}$	Sq. err.	0.8623	0.1582	0.8515	0.0306
			Ab. err.	0.7845	0.1189	0.7699	0.0385
			LINEX ($a^* = 0.1$)	0.6922	0.1794	0.6773	0.0371
			LINEX ($a^* = 0.5$)	0.6190	0.1868	0.6026	0.0406
			LINEX ($a^* = 1.0$)	0.4948	0.2144	0.4889	0.0490
		$Y_{U(3)}$	Sq. err.	1.6310	0.1703	1.6965	0.0742
			Ab. err.	1.5575	0.1975	1.6249	0.0775
			LINEX ($a^* = 0.1$)	1.4980	0.1988	1.5747	0.0759
			LINEX ($a^* = 0.5$)	1.4483	0.2108	1.5294	0.0773
			LINEX ($a^* = 1.0$)	1.3832	0.2281	1.4717	0.0797
		$Y_{U(6)}$	Sq. err.	2.4261	0.2266	2.4843	0.1458
			Ab. err.	2.3270	0.2458	2.4038	0.1300
			LINEX ($a^* = 0.1$)	2.2916	0.2506	2.3806	0.1280
			LINEX ($a^* = 0.5$)	2.2417	0.2660	2.3434	0.1227
			LINEX ($a^* = 1.0$)	2.1699	0.3138	2.2972	0.1170
12	9	$Y_{U(1)}$	Sq. err.	0.9323	0.1145	0.8974	0.0314
			Ab. err.	0.8662	0.1324	0.8328	0.0365
			LINEX ($a^* = 0.1$)	0.7744	0.1246	0.7403	0.0344
			LINEX ($a^* = 0.5$)	0.7066	0.1295	0.6724	0.0363
			LINEX ($a^* = 1.0$)	0.6062	0.1392	0.4571	0.0408
		$Y_{U(5)}$	Sq. err.	2.2125	0.2054	2.2037	0.1157
			Ab. err.	2.1452	0.2247	2.1411	0.1123
			LINEX ($a^* = 0.1$)	2.1106	0.2261	2.1075	0.1112
			LINEX ($a^* = 0.5$)	2.0737	0.2343	2.0724	0.1101
			LINEX ($a^* = 1.0$)	2.0274	0.2453	2.0283	0.1092
		$Y_{U(9)}$	Sq. err.	2.9784	0.3093	3.1510	0.1147
			Ab. err.	2.9075	0.3050	3.0744	0.1148
			LINEX ($a^* = 0.1$)	2.8894	0.3077	3.0550	0.1130
			LINEX ($a^* = 0.5$)	2.8575	0.3085	3.0206	0.1130
			LINEX ($a^* = 1.0$)	2.8179	0.3107	2.9779	0.1133

Comments and observations

From Tables 5.13-5.16, we may observe the following remarks:

(1) It is observed that from Tables 5.13 and 5.15, the predicted values for the future records $X_{U(n)}$ (unobserved records $X_{U(k)}$) under different loss functions, are quite close to each other and fall in their corresponding 95% prediction intervals, based on one-sample and two-sample prediction problems.

(2) From Tables 5.14 and 5.16, we notice that the predictors of the future record $X_{U(n)}$ (unobserved records $X_{U(k)}$) obtained by using Prior 1 perform better than the predictors obtained by using Prior 0.

Example 2: (real data)

In this example we analyze the total seasonal annual rainfall (in inches) recorded at Loss Angeles Civic Center during 132 years, from 1878 to 2009 (season July 1 - June 30). The data set can be obtained from the loss Angeles Civic Website: http://www.laalmanac.com/weather/we13.htm.
For the complete data set, we have computed the maximum likelihood estimators of α and λ and they are 2.2438 and 0.0018, respectively. The corresponding Kolmogorov-Smirnov (KS) distance becomes 0.0939 and the associated p-value is 0.1949. Therefore the KS indicates that Weibull distribution can be used to analyze this rainfall data. We used the upper records from 1930 to 2009 which were as follows: 12.54, 16.93, 21.66, 22.41, 23.43, 32.76, 33.44, 37.96.
We compute the Bayes estimates with respect to different loss function: squared error (Sq. err.), absolute error (Abs. err.) and LINEX function with different choices of $a^* : 0.1, 1.0, 5.0$. The results are presented in Table 5.17. All the estimates are quite close to each other. We obtain the 95% credible intervals for α and λ and they are $(1.7187, 1.8592)$ and $(0.0045, 0.0206)$, respectively. It is noticed that the Bayes estimates of α and λ are falling in the credible intervals.
Also, we consider the prediction of the 9-th, 10-th and 11-th future records. The prediction values and the 95% predictive interval of the 9-th, 10-th and 11-th future records are presented in Table 5.18. It is observed that all predicted values, with respect to different loss functions, are all ordered and fall in their corresponding predictive intervals.

Table 5.17: Bayes estimates with respect to different loss functions for the data in example 2.

	Sq. err. Bayes 1	Abs. err. Bayes 2	$a^* = 0.1$ Bayes 3	$a^* = 1.0$ Bayes 4	$a^* = 5$ Bayes 5
α	1.8160	1.8249	1.8155	1.8151	1.8134
λ	0.0110	0.0103	0.0101	0.0094	0.0060

Table 5.18: Point predictors and PIs for the 9-th, 10-th and 11-th future records.

Size of observed sample	$X_{U(n)}$	Loss function	Predicted values	95%PIs
$m = 8$	$X_{U(9)}$	Sq. err.	42.76	$(38.07, 57.32)$
		Ab. err.	41.05	$(38.06, 57.05)$
		LINEX ($a^* = 0.1$)	42.45	$(38.07, 53.48)$
		LINEX ($a^* = 0.5$)	42.35	$(38.07, 52.31)$
		LINEX ($a^* = 1.0$)	42.23	$(38.07, 51.04)$
	$X_{U(10)}$	Sq. err.	47.56	$(38.96, 68.48)$
		Ab. err.	45.44	$(38.88, 67.92)$
		LINEX ($a^* = 0.1$)	46.94	$(38.96, 61.07)$
		LINEX ($a^* = 0.5$)	46.74	$(38.96, 58.87)$
		LINEX ($a^* = 1.0$)	46.51	$(38.96, 56.61)$
	$X_{U(11)}$	Sq. err.	52.36	$(40.44, 78.73)$
		Ab. err.	49.88	$(40.22, 77.63)$
		LINEX ($a^* = 0.1$)	51.44	$(40.43, 67.60)$
		LINEX ($a^* = 0.5$)	51.14	$(40.43, 64.35)$
		LINEX ($a^* = 1.0$)	50.78	$(40.42, 61.17)$

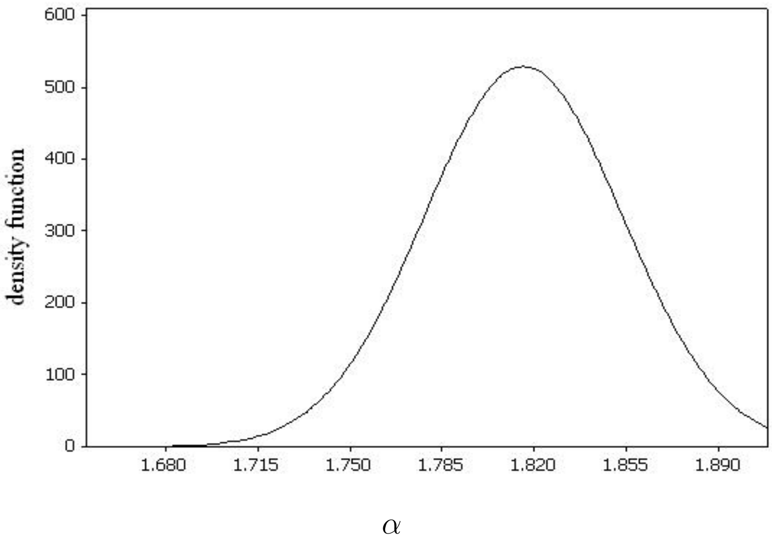

Figure 5.5: Estimate of the density function for α, based on the record data in example 2.

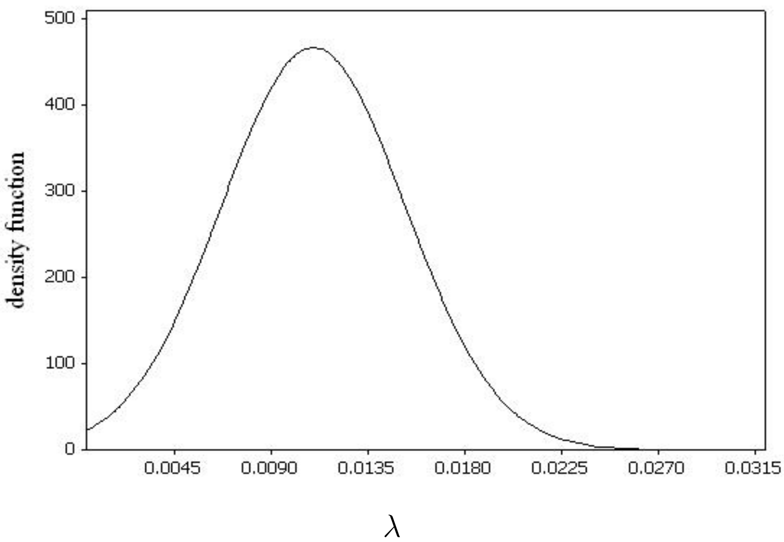

Figure 5.6: Estimate of the density function for λ, based on the record data in example 2.

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Appendix 1

A simple Algorithm for Generating Random Variates with a Log-Concave density, by Devroye (1984):
Let $c = f(m)$, and let f be log-concave on $[m, \infty]$ with mode at m.
Algorithm (Log-Concave densities, Exponential version)

- **Step 0**
  Compute $c \leftarrow f(m)$, $r \leftarrow logc$. (To be done once for each density.)

- **Step 1**
  Generate U uniform on [0,2], and E independent of U and exponential.
  If $U \leq 1$ then $X \leftarrow U$, $T \leftarrow -E$
  else $X \leftarrow 1 + E^*$, $T \leftarrow -E - E^*$(E^* is a new exponential random variate)

- **Step 2**
  $X \leftarrow m + \frac{X}{c}$
  If $T \leq logf(X) - r$ then exit else go to 1

Appendix 2

(*..........ComputationsofMLEsandBayesestimatorsunderdifferentlossfunctions whenbothαandλareunknownbasedonprior0......................*)

$\alpha = 2; \lambda = 1;$ (* The assumed values of the weibull parameters*)

$a = 0; b = 0;$ (*Assumedvaluesofhyperparametersof$\pi 1(\lambda/\text{data})$...Prior0...*)

$c = 0; d = 0;$ (*Assumedvaluesofhyperparametersof$\pi 2(\alpha)$........Prior0...*)

it = 1000; (* number of MCMC samples*)

$M = 100;$ (* number of iterations*)

$n = 30;$ (* size of sample drawn from weibull *)

$m = 10;$ (* size of progressive type 2 censored sample *)

$\text{it1} = m + 5;$

astar1 = 0.1; astar2 = 1; astar3 = 5;

$r[1] = 5; r[2] = 5; r[3] = 5; r[4] = 5; r[5] = 0; r[6] = 0; r[7] = 0; r[8] = 0; r[9] = 0; r[10] = 0;$
$r[11] = 0; r[12] = 0; r[13] = 0; r[14] = 0; r[15] = 0; r[16] = 0; r[17] = 0; r[18] = 0; r[19] = 0;$
$r[20] = 0;$ (* censoring scheme *)

For[$l = 1, l \leq M, l{+}{+}$, (* loop of iterations *)

(*...........Thefollowing loopstogenerateprogressivetype2censoreddata......*)

Do $\left[w[i] = \text{Random}[]; v[i] = w[i]\hat{}\left(1/\left(i + \sum_{k=m-i+1}^{m} r[k]\right)\right), \{i, 1, m\}\right];$

Do $\left[u[i] = 1 - \prod_{k=m-i+1}^{m} v[k]; x[i] = ((-1/\lambda) * \text{Log}[1 - u[i]])\hat{}(1/\alpha), \{i, 1, m\}\right];$

(*......Endofgenerationprogressivedata..*)

(*computationsofiterationsofMLEsforαandλ.Hereshapeparameterαisunknown*)

alphaMLE1[l] =
a1/.
FindRoot $[(m/\text{a1}) + (\sum_{i=1}^{m} \text{Log}[x[i]]) - (m/(\sum_{i=1}^{m}((x[i]\hat{}\text{a1}) * (1 + r[i]))))$
$* (\sum_{i=1}^{m}((x[i]\hat{}\text{a1}) * \text{Log}[x[i]] * (1 + r[i]))) == 0, \{\text{a1}, 2\}];$

lambdaMLE1[l] $= m/(\sum_{i=1}^{m}(x[i]\hat{}\text{alphaMLE1}[l]) * (1 + r[i]));$

(*....computingtheiterationmode...*)

mode =
a2/.
FindRoot $[-d + ((c + m - 1)/\text{a2}) + (\sum_{i=1}^{m} \text{Log}[x[i]]) - ((a + m)/(b + \sum_{i=1}^{m}(x[i]\hat{}\text{a2}) * (1 + r[i])))$
$* (\sum_{i=1}^{m}((x[i]\hat{}\text{a2}) * \text{Log}[x[i]] * (1 + r[i]))) == 0, \{\text{a2}, 2\}];$

fmode =
Log $[e^{-d*\text{mode}} * (\text{mode}\hat{}(c + m - 1)) * (\prod_{i=1}^{m}(x[i]\hat{}(\text{mode} - 1)))$
$* (1/((b + \sum_{i=1}^{m}(x[i]\hat{}\text{mode}) * (1 + r[i]))\hat{}(a + m)))];$

(*....................generatingMCMCsamples$\{(\alpha\text{i}, \lambda\text{i}), i = 1, 2, ..., M = \text{it}\}$..........*)

For[$j = 1, j \leq \text{it}, j{+}{+}$, (* loop of iterations *)

(*...........Generating$\alpha$from$\pi 2(\alpha/\text{data})$byusingDevroye(1984)method.........*)

$\theta = 1;$

$\mathrm{Do}[\mathrm{u2}[j,i] = 2 * \mathrm{Random}[], \{i, 1, \mathrm{it1}\}];$

$\mathrm{Do}[\mathrm{u3}[j,i] = \mathrm{Random}[], \{i, 1, \mathrm{it1}\}];$

$\mathrm{Do}[\mathrm{u4}[j,i] = \mathrm{Random}[], \{i, 1, \mathrm{it1}\}];$

$\mathrm{Do}[e[j,i] = (-1/\theta) * \mathrm{Log}[1 - \mathrm{u3}[j,i]], \{i, 1, \mathrm{it1}\}];$

$\mathrm{Do}[\mathrm{e1}[j,i] = (-1/\theta) * \mathrm{Log}[1 - \mathrm{u4}[j,i]], \{i, 1, \mathrm{it1}\}];$

$\mathrm{Do}[\mathrm{If}[\mathrm{u2}[j,i] \leq 1, \{\mathrm{xx}[j,i] = \mathrm{u2}[j,i], t[j,i] = -e[j,i]\}, \{\mathrm{xx}[j,i] = 1 + \mathrm{e1}[j,i],$
$t[j,i] = -e[j,i] - \mathrm{e1}[j,i]\}], \{i, 1, \mathrm{it1}\}];$

$\mathrm{Do}[\mathrm{x2}[j,i] = \mathrm{mode} + (\mathrm{xx}[j,i]/\mathrm{fmode}), \{i, 1, \mathrm{it1}\}];$

$\mathrm{Do}\left[\mathrm{fx2}[j,i] = \mathrm{Log}\left[e^{-d*\mathrm{x2}[j,i]} * (\mathrm{x2}[j,i]\text{^}(c + m - 1)) * (\prod_{i=1}^{m}((x[i])\text{^}(\mathrm{x2}[j,i] - 1)))\right.\right.$
$\left.\left. * (1/(b + (\sum_{i=1}^{m}(x[i]\text{^}\mathrm{x2}[j,i]) * (1 + r[i]))\text{^}(a + m)))\right], \{i, 1, \mathrm{it1}\}\right];$

$\mathrm{Do}[\mathrm{If}[t[j,i] \leq (\mathrm{Log}[\mathrm{fx2}[j,i]/\mathrm{fmode}]), \{\mathrm{rv}[j,i] = \mathrm{x2}[j,i]\}, \{\mathrm{rv}[j,i] = 222222\}], \{i, 1, \mathrm{it1}\}];$

$\mathrm{Do}[\mathrm{If}[\mathrm{rv}[j,k] \neq 222222, \{\mathrm{alpha}[j] = \mathrm{rv}[j,k], \mathrm{Break}[]\}, \mathrm{alpha}[j] = \mathrm{rv}[j, k+1]], \{k, 1, \mathrm{it1}\}];$

(*...............generatingλfromπ1(λ/α, data)..*)

$\mu = m + a; \sigma = b + \sum_{i=1}^{m}((x[i]\text{^}\mathrm{alpha}[j]) * (1 + r[i]));$

$\mathrm{dist} = \mathrm{GammaDistribution}[\mu, (1/\sigma)];$

$\mathrm{lambda}[j] = \mathrm{Random}[\mathrm{dist}];$

(*..................Endofgeneration..*)

]; (* End of j loop *)

(*...........ComputationofiterationBayesestimators.................................*)

$\mathrm{alphaBE11}[l] = \left(\sum_{i=1}^{\mathrm{it}} \mathrm{alpha}[i]\right) \big/ \mathrm{it};$

$\mathrm{lambdaBE11}[l] = \left(\sum_{i=1}^{\mathrm{it}} \mathrm{lambda}[i]\right) \big/ \mathrm{it};$

$\mathrm{data}\alpha = \mathrm{Table}[\mathrm{alpha}[i], \{i, 1, \mathrm{it}\}];$

$\mathrm{order}\alpha = \mathrm{Sort}[\mathrm{data}\alpha];$

$\mathrm{data}\lambda = \mathrm{Table}[\mathrm{lambda}[i], \{i, 1, \mathrm{it}\}];$

$\mathrm{order}\lambda = \mathrm{Sort}[\mathrm{data}\lambda];$

$\mathrm{alphaBE22}[l] = (\mathrm{order}\alpha[[\mathrm{it}/2]] + \mathrm{order}\alpha[[(\mathrm{it}/2) + 1]])/2;$

$\mathrm{lambdaBE22}[l] = (\mathrm{order}\lambda[[\mathrm{it}/2]] + \mathrm{order}\lambda[[(\mathrm{it}/2) + 1]])/2;$

$\mathrm{alphaBE331}[l] = \left((1/\mathrm{it}) * \sum_{i=1}^{\mathrm{it}}(1/(\mathrm{alpha}[i])\text{^}\mathrm{astar1})\right)\text{^}(-1/\mathrm{astar1});$

$\mathrm{lambdaBE331}[l] = \left((1/\mathrm{it}) * \sum_{i=1}^{\mathrm{it}}(1/(\mathrm{lambda}[i])\text{^}\mathrm{astar1})\right)\text{^}(-1/\mathrm{astar1});$

$\mathrm{alphaBE332}[l] = \left((1/\mathrm{it}) * \sum_{i=1}^{\mathrm{it}}(1/(\mathrm{alpha}[i])\text{^}\mathrm{astar2})\right)\text{^}(-1/\mathrm{astar2});$

$\mathrm{lambdaBE332}[l] = \left((1/\mathrm{it}) * \sum_{i=1}^{\mathrm{it}}(1/(\mathrm{lambda}[i])\text{^}\mathrm{astar2})\right)\text{^}(-1/\mathrm{astar2});$

$\mathrm{alphaBE333}[l] = \left((1/\mathrm{it}) * \sum_{i=1}^{\mathrm{it}}(1/(\mathrm{alpha}[i])\text{^}\mathrm{astar3})\right)\text{^}(-1/\mathrm{astar3});$

$\mathrm{lambdaBE333}[l] = \left((1/\mathrm{it}) * \sum_{i=1}^{\mathrm{it}}(1/(\mathrm{lambda}[i])\text{^}\mathrm{astar3})\right)\text{^}(-1/\mathrm{astar3});$

]; (*...Endof$l$loop....*)

(*computations of MLEs and Bayes estimators and their MSE*)

(*..........MLEs..*)
alphaMLE = $\left(\sum_{l=1}^{M} \text{alphaMLE1}[l]\right) \Big/ M$;
MSE1 = $\left(\sum_{l=1}^{M} (\text{alphaMLE1}[l] - \text{alphaMLE})\hat{}2\right) \Big/ M$;
lambdaMLE = $\left(\sum_{l=1}^{M} \text{lambdaMLE1}[l]\right) \Big/ M$;
MSE2 = $\left(\sum_{l=1}^{M} (\text{lambdaMLE1}[l] - \text{lambdaMLE})\hat{}2\right) \Big/ M$;
(*Bayes estimator under square error loss function*)
alphaBE1 = $\left(\sum_{l=1}^{M} \text{alphaBE11}[l]\right) \Big/ M$;
MSE3 = $\left(\sum_{l=1}^{M} (\text{alphaBE11}[l] - \text{alphaBE1})\hat{}2\right) \Big/ M$;
lambdaBE1 = $\left(\sum_{l=1}^{M} \text{lambdaBE11}[l]\right) \Big/ M$;
MSE4 = $\left(\sum_{l=1}^{M} (\text{lambdaBE11}[l] - \text{lambdaBE1})\hat{}2\right) \Big/ M$;
(*Bayes estimator under absolute error loss function*)
alphaBE2 = $\left(\sum_{l=1}^{M} \text{alphaBE22}[l]\right) \Big/ M$;
MSE5 = $\left(\sum_{l=1}^{M} (\text{alphaBE22}[l] - \text{alphaBE2})\hat{}2\right) \Big/ M$;
lambdaBE2 = $\left(\sum_{l=1}^{M} \text{lambdaBE22}[l]\right) \Big/ M$;
MSE6 = $\left(\sum_{l=1}^{M} (\text{lambdaBE22}[l] - \text{lambdaBE2})\hat{}2\right) \Big/ M$;
(*Bayes estimator under LINRX loss function*)

(*............... $(a^* = 0.1)$ ....*)
alphaBE31 = $\left(\sum_{l=1}^{M} \text{alphaBE331}[l]\right) \Big/ M$;
MSE7 = $\left(\sum_{l=1}^{M} (\text{alphaBE331}[l] - \text{alphaBE31})\hat{}2\right) \Big/ M$;
lambdaBE31 = $\left(\sum_{l=1}^{M} \text{lambdaBE331}[l]\right) \Big/ M$;
MSE8 = $\left(\sum_{l=1}^{M} (\text{lambdaBE331}[l] - \text{lambdaBE31})\hat{}2\right) \Big/ M$;
(*............. $(a^* = 1)$ ....*)
alphaBE32 = $\left(\sum_{l=1}^{M} \text{alphaBE332}[l]\right) \Big/ M$;
MSE9 = $\left(\sum_{l=1}^{M} (\text{alphaBE332}[l] - \text{alphaBE32})\hat{}2\right) \Big/ M$;
lambdaBE32 = $\left(\sum_{l=1}^{M} \text{lambdaBE332}[l]\right) \Big/ M$;
MSE10 = $\left(\sum_{l=1}^{M} (\text{lambdaBE332}[l] - \text{lambdaBE32})\hat{}2\right) \Big/ M$;
(*............ $(a^* = 5)$ ....*)
alphaBE33 = $\left(\sum_{l=1}^{M} \text{alphaBE333}[l]\right) \Big/ M$;
MSE11 = $\left(\sum_{l=1}^{M} (\text{alphaBE333}[l] - \text{alphaBE33})\hat{}2\right) \Big/ M$;
lambdaBE33 = $\left(\sum_{l=1}^{M} \text{lambdaBE333}[l]\right) \Big/ M$;
MSE12 = $\left(\sum_{l=1}^{M} (\text{lambdaBE333}[l] - \text{lambdaBE33})\hat{}2\right) \Big/ M$;
Print["Prior 0 "];
Print["Scheme n = ", n, " m = ", m];

```
Print["Cencoring scheme is (", r[1], ",", r[2], ",", r[3], ",", r[4], ",", r[5], ",",
r[6], ",", r[7], ",", r[8], ",", r[9], ",", r[10], ")"];
Print ["Parameter | MLE | S.E.L | A.E.L | a*=", astar1, "| a*=", astar2, "| a*=", astar3, "|"] ;

Print[" α & ", NumberForm[alphaMLE, 5], " & ", NumberForm[alphaBE1, 5], " & ",
NumberForm[alphaBE2, 5], " & ", NumberForm[alphaBE31, 5], " & ", NumberForm[alphaBE32, 5],
" & ", NumberForm[alphaBE33, 5]];
Print[" MSE(α) & $(", NumberForm[MSE1, 4], ")$&$(", NumberForm[MSE3, 4], ")$&$(",
NumberForm[MSE5, 4], ")$&$(", NumberForm[MSE7, 4], ")$&$(", NumberForm[MSE9, 4],
")$&$(", NumberForm[MSE11, 4], ")$"];
Print[" λ & ", NumberForm[lambdaMLE, 5], " & ", NumberForm[lambdaBE1, 5], " & ",
NumberForm[lambdaBE2, 5], " & ", NumberForm[lambdaBE31, 5], " & ", NumberForm[lambdaBE32,
" & ", NumberForm[lambdaBE33, 5]];
Print[" MSE(λ) & $(", NumberForm[MSE2, 4], ")$&$(", NumberForm[MSE4, 4], ")$&$(",
NumberForm[MSE6, 4], ")$&$(", NumberForm[MSE8, 4], ")$&$(", NumberForm[MSE10, 4],
")$&$(", NumberForm[MSE12, 4], ")$"];
```

Appendix 3

(* One sample prediction based on progrressive data *)

$\alpha = 2; \lambda = 1;$ (* The assumed values of the parameters weibull *)

$a = 1; b = 1;$ (*Assumedvaluesofhyperparametersofπ1(λ/data)...Prior1..*)

$c = 2; d = 1;$ (*Assumedvaluesofhyperparametersofπ2(α).........Prior1..*)

$\tau = 0.05;$ (* The coverage of the credible interval *)

it = 1000(* number of interations*);

astar1 = 0.1; astar2 = 0.5; astar3 = 1;

$n = 30$(* size of sample drawn from weibull *);

$m = 10$(* size of progressive type 2 censored sample *);

it1 = $m + 5$;

(*================================= = (censoringscheme)=============

$r[1] = 5; r[2] = 5; r[3] = 5; r[4] = 5; r[5] = 0; r[6] = 0; r[7] = 0; r[8] = 0; r[9] = 0; r[10] = 0;$

$r[11] = 0; r[12] = 0; r[13] = 0; r[14] = 0; r[15] = 0; r[16] = 0; r[17] = 0; r[18] = 0; r[19] = 0;$

$r[20] = 0;$ (* censoring scheme *)

(*================== == (generatingdatafromweibul)====================

Do $\left[w[i] = \text{Random}[]; v[i] = w[i]\text{^}\left(1\,/\left(i + \sum_{k=m-i+1}^{m} r[k]\right)\right), \{i, 1, m\}\right];$

Do $\left[u[i] = 1 - \prod_{k=m-i+1}^{m} v[k]; x[i] = ((-1/\lambda) * \text{Log}[1 - u[i]])\text{^}(1/\alpha), \{i, 1, m\}\right];$

Print["The observed progressive type 2 censored sample "];

Do[Print["x", i, ":", m, ":", n, " = ", NumberForm[$x[i]$, 5]], $\{i, 1, m\}$];

Print[" Y k:rj ", " predicted value", " 95% prediction interval"];

Print["==

mode =

a2/.

FindRoot $\left[-d + ((m + c - 1)/\text{a2}) + \left(\sum_{i=1}^{m} \text{Log}[x[i]]\right) - \left((a + m)\,/\left(b + \sum_{i=1}^{m}(x[i]\text{^a2}) * (1 + r[i])\right)\right)\right.$

$\left. * \left(\sum_{i=1}^{m}((x[i]\text{^a2}) * \text{Log}[x[i]] * (1 + r[i]))\right) == 0, \{\text{a2}, 2\}\right];$

fmode =

Log $\left[e^{-d*\text{mode}} * (\text{mode}\text{^}(m + c - 1)) * \left(\prod_{i=1}^{m}(x[i]\text{^}(\text{mode} - 1))\right) *\right.$

$\left.\left(1\,/\left(\left(b + \sum_{i=1}^{m}(x[i]\text{^mode}) * (1 + r[i])\right)\text{^}(a + m)\right)\right)\right];$

(*..........One − SamplePrediction...*)

(*..........Computationsofpredictionestimators...........................*)

For[j1 = 1, j1 $\leq m$, j1++,

Linitial = x[j1] + 0.05;

Uinitial = Linitial + 0.5;

Minitial = Linitial;

If[r[j1] == 0, Goto[end]];

For[$k = 1, k \leq r$[j1], k++,

CC = ((r[j1])!)/((($k - 1$)!) * ((r[j1] − k)!));

For[$j = 1, j \leq$ it, j++,

(*...........GeneratingMCMCsamples$\{(\alpha\text{i}, \lambda\text{i}), i = 1, ..., M = \text{it}\}$..*)

(*...........Generating$\alpha$from$\pi2(\alpha)$byusingDevroye(1984)method.....................................*)

$\theta = 1;$

$\text{Do}[\text{u2}[j, i] = 2 * \text{Random}[], \{i, 1, \text{it1}\}];$

$\text{Do}[\text{u3}[j, i] = \text{Random}[], \{i, 1, \text{it1}\}];$

$\text{Do}[\text{u4}[j, i] = \text{Random}[], \{i, 1, \text{it1}\}];$

$\text{Do}[e[j, i] = (-1/\theta) * \text{Log}[1 - \text{u3}[j, i]], \{i, 1, \text{it1}\}];$

$\text{Do}[\text{e1}[j, i] = (-1/\theta) * \text{Log}[1 - \text{u4}[j, i]], \{i, 1, \text{it1}\}];$

$\text{Do}[\text{If}[\text{u2}[j, i] \leq 1, \{\text{xx}[j, i] = \text{u2}[j, i], t[j, i] = -e[j, i]\}, \{\text{xx}[j, i] = 1 + \text{e1}[j, i],$
$t[j, i] = -e[j, i] - \text{e1}[j, i]\}], \{i, 1, \text{it1}\}];$

$\text{Do}[\text{x2}[j, i] = \text{mode} + (\text{xx}[j, i]/\text{fmode}), \{i, 1, \text{it1}\}];$

$\text{Do}\left[\text{fx2}[j, i] = \text{Log}\left[e^{-d*\text{x2}[j,i]} * (\text{x2}[j, i]\wedge(m + c - 1)) * (\prod_{i=1}^{m}((x[i])\wedge(\text{x2}[j, i] - 1))) *\right.\right.$
$\left.\left.(1 / ((b + \sum_{i=1}^{m}(x[i]\wedge\text{x2}[j, i]) * (1 + r[i]))\wedge(a + m)))\right], \{i, 1, \text{it1}\}\right];$

$\text{Do}[\text{If}[t[j, i] \leq (\text{Log}[\text{fx2}[j, i]/\text{fmode}]), \{\text{rv}[j, i] = \text{x2}[j, i]\}, \{\text{rv}[j, i] = 222222\}], \{i, 1, \text{it1}\}];$

$\text{Do}[\text{If}[\text{rv}[j, k] \neq 222222, \{\text{alpha}[j] = \text{rv}[j, k], \text{Break}[]\}, \text{alpha}[j] = \text{rv}[j, k + 1]], \{k, 1, \text{it1}\}];$

(*......enerating$\lambda$from$\pi1(\lambda/\text{data})$............................*)

$\mu = m + a; \sigma = b + \sum_{i=1}^{m}((x[i]\wedge\text{alpha}[j]) * (1 + r[i]));$

$\text{dist} = \text{GammaDistribution}[\mu, (1/\sigma)];$

$\text{lambda}[j] = \text{Random}[\text{dist}];$

(*......Endofgeneration..*)

$\text{sum1}[j] = \text{CC} * \left(\sum_{i=0}^{k-1}\left(\text{Binomial}[k - 1, i] * ((-1)\wedge(k - i - 1)) * e^{(-\text{lambda}[j]*(i - r[\text{j1}])*(x[\text{j1}]\wedge\text{alpha}[j]))} *\right.\right.$
$(\text{Gamma}[(1/\text{alpha}[j]) + 1, \text{lambda}[j] * (r[\text{j1}] - i) * (x[\text{j1}]\wedge\text{alpha}[j])]/$
$\left.\left.(\text{lambda}[j]\wedge(1/\text{alpha}[j]) * ((r[\text{j1}] - i)\wedge((1/\text{alpha}[j]) + 1))))\right)\right);$

$\text{sum2}[j] = \text{CC} * \left(\sum_{i=0}^{k-1}\left(\text{Binomial}[k - 1, i] * ((-1)\wedge(k - i - 1)) * e^{(-\text{lambda}[j]*(i - r[\text{j1}])*(x[\text{j1}]\wedge\text{alpha}[j]))} *\right.\right.$
$\left.\left.\left(e^{(-\text{lambda}[j]*(r[\text{j1}] - i)*(y\wedge\text{alpha}[j]))} / (r[\text{j1}] - i)\right)\right)\right);$

$\text{sum31}[j] = \text{CC} * \left(\sum_{i=0}^{k-1}\left(\text{Binomial}[k - 1, i] * ((-1)\wedge(k - i - 1)) * e^{(-\text{lambda}[j]*(i - r[\text{j1}])*(x[\text{j1}]\wedge\text{alpha}[j]))} *\right.\right.$
$(\text{Gamma}[1 - (\text{astar1}/\text{alpha}[j]), \text{lambda}[j] * (r[\text{j1}] - i) * (x[\text{j1}]\wedge\text{alpha}[j])]/$
$\left.\left.((\text{lambda}[j]\wedge(-\text{astar1}/\text{alpha}[j])) * ((r[\text{j1}] - i)\wedge(1 - (\text{astar1}/\text{alpha}[j])))))\right)\right);$

$\text{sum32}[j] = \text{CC} * \left(\sum_{i=0}^{k-1}\left(\text{Binomial}[k - 1, i] * ((-1)\wedge(k - i - 1)) * e^{(-\text{lambda}[j]*(i - r[\text{j1}])*(x[\text{j1}]\wedge\text{alpha}[j]))} *\right.\right.$
$(\text{Gamma}[1 - (\text{astar2}/\text{alpha}[j]), \text{lambda}[j] * (r[\text{j1}] - i) * (x[\text{j1}]\wedge\text{alpha}[j])]/$
$\left.\left.((\text{lambda}[j]\wedge(-\text{astar2}/\text{alpha}[j])) * ((r[\text{j1}] - i)\wedge(1 - (\text{astar2}/\text{alpha}[j])))))\right)\right);$

$\text{sum33}[j] = \text{CC} * \left(\sum_{i=0}^{k-1}\left(\text{Binomial}[k - 1, i] * ((-1)\wedge(k - i - 1)) * e^{(-\text{lambda}[j]*(i - r[\text{j1}])*(x[\text{j1}]\wedge\text{alpha}[j]))} *\right.\right.$
$(\text{Gamma}[1 - (\text{astar3}/\text{alpha}[j]), \text{lambda}[j] * (r[\text{j1}] - i) * (x[\text{j1}]\wedge\text{alpha}[j])]/$
$\left.\left.((\text{lambda}[j]\wedge(-\text{astar3}/\text{alpha}[j])) * ((r[\text{j1}] - i)\wedge(1 - (\text{astar3}/\text{alpha}[j])))))\right)\right);$

$\text{Survival}[j] = \text{CC} * \left(\sum_{i=0}^{k-1} \left(\text{Binomial}[k-1,i] * ((-1)\verb|^|(k-i-1)) * e^{-\text{lambda}[j]*(i-r[\text{j1}])*(x[\text{j1}]\verb|^|\text{alpha}[j])} * \left(e^{-\text{lambda}[j]*(r[\text{j1}]-i)*(z\verb|^|\text{alpha}[j])} \middle/ (r[\text{j1}]-i)\right)\right)\right);$

];

(*........Undersquareerrorlossfunction ....*)

$\text{yBP1} = \left(\sum_{j=1}^{\text{it}} \text{sum1}[j]\right) \Big/ \text{it};$

$\text{PSurvival1} = \left(\sum_{j=1}^{\text{it}} \text{Survival}[j]\right) \Big/ \text{it};$

Lower1 =

$z/.$

$\text{FindRoot}[\text{PSurvival1} == (1-(\tau/2)), \{z, \text{Linitial}\}];$

Upper1 =

$z/.$

$\text{FindRoot}[\text{PSurvival1} == (\tau/2), \{z, \text{Uinitial}\}];$

(*........Underabsoluteerrorlossfunction ......*)

$\text{sum} = \left(\sum_{j=1}^{\text{it}} \text{sum2}[j]\right) \Big/ \text{it};$

yBP2 =

$y/.$

$\text{FindRoot}[\text{sum} == 0.5, \{y, \text{Minitial}\}];$

$\text{Survivals} = \text{Table}[\text{Survival}[j], \{j, 1, \text{it}\}];$

$\text{sort} = \text{Sort}[\text{Survivals}];$

$\text{PSurvival2} = (\text{sort}[[\text{it}/2]] + \text{sort}[[(\text{it}/2)+1]])/2;$

Lower2 =

$z/.$

$\text{FindRoot}[\text{PSurvival2} == (1-(\tau/2)), \{z, \text{Linitial}\}];$

Upper2 =

$z/.$

$\text{FindRoot}[\text{PSurvival2} == (\tau/2), \{z, \text{Uinitial}\}];$

(*.... UnderLINEXlossfunction...........*)

(*........... $(a^* = 0.1)$ ....*)

$\text{yBP31} = \left((1/\text{it}) * \left(\sum_{j=1}^{\text{it}} \text{sum31}[j]\right)\right)\verb|^|(-1/\text{astar1});$

$\text{PSurvival31} = \left(\left(\sum_{j=1}^{\text{it}} (\text{Survival}[j]\verb|^|(-\text{astar1}))\right) \Big/ \text{it}\right)\verb|^|(-1/\text{astar1});$

Lower31 =

$z/.$

$\text{FindRoot}[\text{PSurvival31} == (1-(\tau/2)), \{z, \text{Linitial}\}];$

Upper31 =

```
z/.
FindRoot[PSurvival31 == (τ/2), {z, Uinitial}];
(*............ (a* = 0.5) ......................*)
yBP32 = ((1/it) * (Sum[sum32[j], {j, 1, it}]))^(-1/astar2);
PSurvival32 = ((Sum[(Survival[j]^(-astar2)), {j, 1, it}]) / it)^(-1/astar2);
Lower32 =
z/.
FindRoot[PSurvival32 == (1 - (τ/2)), {z, Linitial}];
Upper32 =
z/.
FindRoot[PSurvival32 == (τ/2), {z, Uinitial}];
(*............. (a* = 1) .......................*)
yBP33 = ((1/it) * (Sum[sum33[j], {j, 1, it}]))^(-1/astar3);
PSurvival33 = ((Sum[(Survival[j]^(-astar3)), {j, 1, it}]) / it)^(-1/astar3);
Lower33 =
z/.
FindRoot[PSurvival33 == (1 - (τ/2)), {z, Linitial}];
Upper33 =
z/.
FindRoot[PSurvival33 == (τ/2), {z, Uinitial}];
(*.............................................*)
Print[" Y", k, ":r", j1];
Print["================="];
Print["\\multirow{1}{*}{}&&&", "Sq. err. &", NumberForm[Re[yBP1], 4],
" &$(", NumberForm[Re[Lower1], 4], ",", NumberForm[Re[Upper1], 4], ")$\\\\"];
Print["\\multirow{1}{*}{}&&&", "Ab. err. &", NumberForm[Re[yBP2], 4],
" &$(", NumberForm[Re[Lower2], 4], ",", NumberForm[Re[Upper2], 4], ")$\\\\"];
Print ["\\multirow{1}{*}{}&&&", "LINEX (a*=0.1) &", NumberForm[Re[yBP31], 4],
" &$(", NumberForm[Re[Lower31], 4], ",", NumberForm[Re[Upper31], 4], ")$\\\\"];
Print ["\\multirow{1}{*}{}&&&", "LINEX (a*=0.5) &", NumberForm[Re[yBP32], 4],
" &$(", NumberForm[Re[Lower32], 4], ",", NumberForm[Re[Upper32], 4], ")$\\\\"];
Print ["\\multirow{1}{*}{}&&&", "LINEX (a*=1.0) &", NumberForm[Re[yBP33], 4],
" &$(", NumberForm[Re[Lower33], 4], ",", NumberForm[Re[Upper33], 4], ")$\\\\ \cline{3-6}"];
Linitial = Lower1 + 0.05;
Uinitial = Upper1 + 0.05;
```

```
Minitial = yBP2 + 0.05;
];
Label[end];
];
```

Appendix 4

(* Two samples prediction based on progressive type II data *)

$\alpha = 2; \lambda = 1;$ (* The assumed values of the parameters weibull *)

$a = 1; b = 1;$ (*Assumedvaluesofhyperparametersof$\pi 1(\lambda/\text{data})$..Prior1..*)

$c = 2; d = 1;$ (*Assumedvaluesofhyperparametersof$\pi 2(\alpha)$.......Prior1...*)

$\tau = 0.05;$ (* The coverage of the credible interval *)

it = 1000(* number of interations*);

astar1 = 0.1; astar2 = 0.5; astar3 = 1;

n1 = 30(* size of observed sample drawn from weibull *);

m1 = 10(* size of progressive type 2 censored observed sample *);

it1 = m1 + 5;

$r[1] = 5; r[2] = 5; r[3] = 5; r[4] = 5; r[5] = 0; r[6] = 0; r[7] = 0; r[8] = 0; r[9] = 0; r[10] = 0;$

$r[11] = 0; r[12] = 0; r[13] = 0; r[14] = 0; r[15] = 0;$ (* censoring scheme for observed sample *)

(*=============== = (generatingprogressivedatafromweibull)=========*)

$\text{Do}\left[w[i] = \text{Random}[]; v[i] = w[i]\hat{}\left(1/\left(i + \sum_{\text{k1}=\text{m1}-i+1}^{\text{m1}} r[\text{k1}]\right)\right), \{i, 1, \text{m1}\}\right];$

$\text{Do}\left[u[i] = 1 - \prod_{\text{k2}=\text{m1}-i+1}^{\text{m1}} v[\text{k2}]; x[i] = ((-1/\lambda) * \text{Log}[1 - u[i]])\hat{}(1/\alpha), \{i, 1, \text{m1}\}\right];$

(*==

Print["The observed progressive type 2 censored sample "];

Do[Print["x", i, ":", m1, ":", n1, " = ", $x[i]$], $\{i, 1, \text{m1}\}$];

n2 = 10; (* size of unobserved sample drawn from weibull *)

m2 = 6; (* size of progressive type 2 censored future sample *);

$s[1] = 4; s[2] = 0; s[3] = 0; s[4] = 0; s[5] = 0; s[6] = 0;$ (* censoring scheme for future sample *)

mode =

a2/.

$\text{FindRoot}\left[-d + ((c + \text{m1} - 1)/\text{a2}) + \left(\sum_{i=1}^{\text{m1}} \text{Log}[x[i]]\right) - \left((a + \text{m1})/\left(b + \sum_{i=1}^{\text{m1}} (x[i]\hat{}\text{a2}) * (1 + r[i])\right)\right)\right.$
$\left. * \left(\sum_{i=1}^{\text{m1}} ((x[i]\hat{}\text{a2}) * \text{Log}[x[i]] * (1 + r[i]))\right) == 0, \{\text{a2}, 2\}\right];$

fmode =

$\text{Log}\left[e^{-d*\text{mode}} * (\text{mode}\hat{}(c + \text{m1} - 1)) * \left(\prod_{i=1}^{\text{m1}} (x[i]\hat{}(\text{mode} - 1))\right) *\right.$
$\left.\left(1/\left(\left(b + \sum_{i=1}^{\text{m1}} (x[i]\hat{}\text{mode}) * (1 + r[i])\right)\hat{}(a + \text{m1})\right)\right)\right];$

For[$j = 1, j \leq \text{it}, j$++,

$\theta = 1;$

Do[u2[j, i] = 2 * Random[], $\{i, 1, \text{it1}\}$];

Do[u3[j, i] = Random[], $\{i, 1, \text{it1}\}$];

Do[u4[j, i] = Random[], $\{i, 1, \text{it1}\}$];

Do[$e[j, i] = (-1/\theta) * \text{Log}[1 - \text{u3}[j, i]], \{i, 1, \text{it1}\}$];

Do[$\text{e1}[j, i] = (-1/\theta) * \text{Log}[1 - \text{u4}[j, i]], \{i, 1, \text{it1}\}$];

Do[If[$\text{u2}[j, i] \leq 1, \{\text{xx}[j, i] = \text{u2}[j, i], t[j, i] = -e[j, i]\}, \{\text{xx}[j, i] = 1 + \text{e1}[j, i],$

$t[j, i] = -e[j, i] - \text{e1}[j, i]\}], \{i, 1, \text{it1}\}$];

Do[$\text{x2}[j, i] = \text{mode} + (\text{xx}[j, i]/\text{fmode}), \{i, 1, \text{it1}\}$];

$\text{Do}\left[\text{fx2}[j,i]=\text{Log}\left[e^{-d*\text{x2}[j,i]}*(\text{x2}[j,i]\text{^}(c+\text{m1}-1))*\left(\prod_{i=1}^{\text{m1}}((x[i])\text{^}(\text{x2}[j,i]-1))\right)\right.\right.$
$\left.\left.*\left(1/\left(\left(b+\sum_{i=1}^{\text{m1}}(x[i]\text{^}\text{x2}[j,i])*(1+r[i])\right)\text{^}(a+\text{m1})\right)\right)\right],\{i,1,\text{it1}\}\right];$
$\text{Do}[\text{If}[t[j,i]\leq(\text{Log}[\text{fx2}[j,i]/\text{fmode}]),\{\text{rv}[j,i]=\text{x2}[j,i]\},\{\text{rv}[j,i]=222222\}],\{i,1,\text{it1}\}];$
$\text{Do}[\text{If}[\text{rv}[j,k]\neq 222222,\{\text{alpha}[j]=\text{rv}[j,k],\text{Break}[]\},\text{alpha}[j]=\text{rv}[j,k+1]],\{k,1,\text{it1}\}];$
$\mu=\text{m1}+a;\sigma=b+\sum_{i=1}^{\text{m1}}((x[i]\text{^}\text{alpha}[j])*(1+r[i]));$
$\text{dist}=\text{GammaDistribution}[\mu,(1/\sigma)];$
$\text{lambda}[j]=\text{Random}[\text{dist}];$
$];$
$\text{Minitial}=x[1];$
$\text{Linitial}=x[1];$
$\text{Uinitial}=\text{Linitial}+0.5;$
$\text{For}[k=1,k\leq\text{m2},k\text{++},$
$\gamma[1]=\text{n2};$
$\text{Do}\left[\gamma[i]=\left(\text{n2}-\left(\sum_{j=1}^{i-1}s[j]\right)-i+1\right),\{i,2,\text{m2}\}\right];$
$\text{Ckminus1}=\left(\prod_{i=1}^{k}\gamma[i]\right);$
$\text{If}[k\text{==}1,\text{aik}[1]=1];$
$\text{If}[k\text{==}2,\text{aik}[1]=(1/(\gamma[2]-\gamma[1]));\text{aik}[2]=(1/(\gamma[1]-\gamma[2]))];$
$\text{If}\left[k>2,\text{aik}[1]=\left(\prod_{j=2}^{k}(1/(\gamma[j]-\gamma[1]))\right);\right.$
$\text{Do}\left[\text{prod1}=\left(\prod_{j=1}^{i-1}(1/(\gamma[j]-\gamma[i]))\right);\text{prod2}=\left(\prod_{j=i+1}^{k}(1/(\gamma[j]-\gamma[i]))\right);\right.$
$\left.\text{aik}[i]=\text{prod1}*\text{prod2},\{i,2,k-1\}\right];$
$\left.\text{aik}[k]=\left(\prod_{j=1}^{k-1}(1/(\gamma[j]-\gamma[k]))\right)\right];$
$\text{For}[j=1,j\leq\text{it},j\text{++},$
$\text{yBP1}[j]=\sum_{i=1}^{k}(\text{Ckminus1}*\text{aik}[i]*(\text{Gamma}[1+(1/\text{alpha}[j])]/((\text{lambda}[j]\text{^}(1/\text{alpha}[j]))$
$*(\gamma[i]\text{^}(1+(1/\text{alpha}[j]))))));$
$\text{sum}[j]=\sum_{i=1}^{k}\left(\left(\text{Ckminus1}*\text{aik}[i]*e^{-\text{lambda}[j]*\gamma[i]*(y\text{^}\text{alpha}[j])}\right)/\gamma[i]\right);$
$\text{yBP31}[j]=\sum_{i=1}^{k}(\text{Ckminus1}*\text{aik}[i]*(\text{Gamma}[1-(\text{astar1}/\text{alpha}[j])]/((\text{lambda}[j]$
$\text{^}(-\text{astar1}/\text{alpha}[j]))*(\gamma[i]\text{^}(1-(\text{astar1}/\text{alpha}[j]))))));$
$\text{yBP32}[j]=\sum_{i=1}^{k}(\text{Ckminus1}*\text{aik}[i]*(\text{Gamma}[1-(\text{astar2}/\text{alpha}[j])]/((\text{lambda}[j]$
$\text{^}(-\text{astar2}/\text{alpha}[j]))*(\gamma[i]\text{^}(1-(\text{astar2}/\text{alpha}[j]))))));$
$\text{yBP33}[j]=\sum_{i=1}^{k}(\text{Ckminus1}*\text{aik}[i]*(\text{Gamma}[1-(\text{astar3}/\text{alpha}[j])]/((\text{lambda}[j]$
$\text{^}(-\text{astar3}/\text{alpha}[j]))*(\gamma[i]\text{^}(1-(\text{astar3}/\text{alpha}[j]))))));$
$\text{Gky}[j]=\sum_{i=1}^{k}\left(\left(\text{Ckminus1}*\text{aik}[i]*\left(1-e^{-\text{lambda}[j]*\gamma[i]*(y\text{^}\text{alpha}[j])}\right)\right)/\gamma[i]\right);$
$];$
$\text{yBP1hat}=\left(\sum_{j=1}^{\text{it}}\text{yBP1}[j]\right)/\text{it};$
$\text{Gky1hat}=\left(\sum_{j=1}^{\text{it}}\text{Gky}[j]\right)/\text{it};$
$\text{Lower1}=$
$y/.$
$\text{FindRoot}[\text{Gky1hat}==(\tau/2),\{y,\text{Linitial}\}];$

Upper1 =

$y/.$

FindRoot[Gky1hat == (1 − (τ/2)), {y, Uinitial}];

sum1 = $\left(\sum_{j=1}^{\text{it}} \text{sum}[j]\right) \Big/ \text{it}$;

yBP2hat =

$y/.$

FindRoot[sum1 == 0.5, {y, Minitial}];

CDFs = Table[Gky[j], {j, 1, it}];

sort = Sort[CDFs];

PCDF2 = (sort[[it/2]] + sort[[(it/2) + 1]])/2;

Lower2 =

$y/.$

FindRoot[PCDF2 == (τ/2), {y, Linitial}];

Upper2 =

$y/.$

FindRoot[PCDF2 == (1 − (τ/2)), {y, Uinitial}];

yBP31hat = $\left(\left(\sum_{j=1}^{\text{it}} \text{yBP31}[j]\right) \Big/ \text{it}\right)$^(−1/astar1);

Gky31hat = $\left(\left(\sum_{j=1}^{\text{it}} (\text{Gky}[j]\text{\^{}}(-\text{astar1}))\right) \Big/ \text{it}\right)$^(−1/astar1);

Lower31 =

$y/.$

FindRoot[Gky31hat == (τ/2), {y, Linitial}];

Upper31 =

$y/.$

FindRoot[Gky31hat == (1 − (τ/2)), {y, Uinitial}];

yBP32hat = $\left(\left(\sum_{j=1}^{\text{it}} \text{yBP32}[j]\right) \Big/ \text{it}\right)$^(−1/astar2);

Gky32hat = $\left(\left(\sum_{j=1}^{\text{it}} (\text{Gky}[j]\text{\^{}}(-\text{astar2}))\right) \Big/ \text{it}\right)$^(−1/astar2);

Lower32 =

$y/.$

FindRoot[Gky32hat == (τ/2), {y, Linitial}];

Upper32 =

$y/.$

FindRoot[Gky32hat == (1 − (τ/2)), {y, Uinitial}];

yBP33hat = $\left(\left(\sum_{j=1}^{\text{it}} \text{yBP33}[j]\right) \Big/ \text{it}\right)$^(−1/astar3);

Gky33hat = $\left(\left(\sum_{j=1}^{\text{it}} (\text{Gky}[j]\text{\^{}}(-\text{astar3}))\right) \Big/ \text{it}\right)$^(−1/astar3);

Lower33 =

$y/.$

FindRoot[Gky33hat == (τ/2), {y, Linitial}];

Upper33 =

```
y/.
FindRoot[Gky33hat == (1 − (τ/2)), {y, Uinitial}];
Print["\\multirow{1}{*}{}&&$Y_{", k, ":", m2, ":", n2, "}$&", "Sq. err. &",
NumberForm[Re[yBP1hat], 5], " &$(", NumberForm[Re[Lower1], 5],
",", NumberForm[Re[Upper1], 5], ")$\\\\"];
Print["\\multirow{1}{*}{}&&&", "Ab. err. &",
NumberForm[Re[yBP2hat], 5], " &$(", NumberForm[Re[Lower2], 5], ",",
NumberForm[Re[Upper2], 5], ")$\\\\"];
Print ["\\multirow{1}{*}{}&&&", "LINEX $(a*=0.1)$ &",
NumberForm[Re[yBP31hat], 5], " &$(", NumberForm[Re[Lower31], 5], ",",
NumberForm[Re[Upper31], 5], ")$\\\\"];
Print ["\\multirow{1}{*}{}&&&", "LINEX $(a*=0.5)$ &",
NumberForm[Re[yBP32hat], 5], " &$(", NumberForm[Re[Lower32], 5], ",",
NumberForm[Re[Upper32], 5], ")$\\\\"];
Print ["\\multirow{1}{*}{}&&&", "LINEX $(a*=1.0)$ &",
NumberForm[Re[yBP33hat], 5], " &$(", NumberForm[Re[Lower33], 5], ",",
NumberForm[Re[Upper33], 5], ")$\\\\ \cline{3-6}"];
Minitial = yBP2hat + 0.05;
Linitial = Lower1 + 0.05;
Uinitial = Upper1 + 0.05;
];
```

Appendix 5

(* ComputationsofMLEsandBayesestimatorsunderdifferentlossfunctions
whenbothαandλareunknownbasedonrecorddata, byusingprior0 *)
$\alpha = 2; \lambda = 1;$ (* The assumed values of the weibull parameters*)
$a = 0; b = 0;$ (*Assumedvaluesofhyperparametersof$\pi 1(\lambda/\text{data})$...Prior0...*)
$c = 0; d = 0;$ (*Assumedvaluesofhyperparametersof$\pi 2(\alpha)$........Prior0....*)
it = 1000; (* number of MCMC samples*)
$M = 100;$ (* number of iterations *)
$n = 9$(* size of record sample drawn from weibull *);
it1 $= n + 10;$
astar1 = 0.1; astar2 = 1; astar3 = 5;
For[$l = 1, l \leq M, l$++, (* loop of iterations *)
(*====== = (generatingrecorddatafromweibull)========= = *)
Do[$u[i]$ = Random[], $\{i, 1, 100\}$];
$\beta 1 = 1;$
Do[$e[i] = (-1/\beta 1) *$ Log[$1 - u[i]$], $\{i, 1, 100\}$];
Do $\left[x[k] = \left(\left(\sum_{i=1}^{k}(e[i])\right) / \lambda\right) \hat{}(1/\alpha), \{k, 1, n\}\right]$;
(*== == ===(endofgeneration)============================*)
(*....ComuputationtheiterationMLEforαandλ.......*)
alphaMLE1[l] $= n / ((n *$ Log[$x[n]$]) $- \sum_{i=1}^{n}$ Log[$x[i]$]) ;
lambdaMLE1[l] $= n * (x[n]\hat{}(-$alphaMLE1[l]));
mode =
a1/.
FindRoot $[-d + ((c + n - 1)/\text{a1}) + (\sum_{i=1}^{n} \text{Log}[x[i]]) - ((a + n) * (x[n]\hat{}\text{a1})$
$*\text{Log}[x[n]])/(b + x[n]\hat{}\text{a1}) == 0, \{\text{a1}, 2\}]$;
fmode = Log $\left[e^{-d*\text{mode}} * (\text{mode}\hat{}(c + n - 1)) * (\prod_{i=1}^{n}((x[i])\hat{}(\text{mode} - 1)))\right.$
$\left.*(1/(b + (x[n]\hat{}\text{mode}))\hat{}(a + n))\right]$;
(*......generatingMCMCsamples............................*)
For[$j = 1, j \leq$ it, j++, (* loop of iterations *)
(*Generating$\alpha$from$\pi 2(\alpha/\text{data})$byusingDevroye(1984)method*)
$\theta = 1;$
Do[u2[j, i] = 2 * Random[], $\{i, 1, \text{it1}\}$];
Do[u3[j, i] = Random[], $\{i, 1, \text{it1}\}$];
Do[u4[j, i] = Random[], $\{i, 1, \text{it1}\}$];
Do[$e[j, i] = (-1/\theta) *$ Log[1 − u3[j, i]], $\{i, 1, \text{it1}\}$];
Do[e1[j, i] $= (-1/\theta) *$ Log[1 − u4[j, i]], $\{i, 1, \text{it1}\}$];
Do[If[u2[j, i] ≤ 1, {xx[j, i] = u2[j, i], $t[j, i] = -e[j, i]$}, {xx[j, i] = 1 + e1[j, i],
$t[j, i] = -e[j, i] -$ e1[j, i]}], $\{i, 1, \text{it1}\}$];
Do[x2[j, i] = mode + (xx[j, i]/fmode), $\{i, 1, \text{it1}\}$];

$\mathrm{Do}\left[\mathrm{fx2}[j,i]=\mathrm{Log}\left[e^{-d*\mathrm{x2}[j,i]}*((\mathrm{x2}[j,i])\hat{}(c+n-1))*(\prod_{i=1}^{n}((x[i])\hat{}(\mathrm{x2}[j,i]-1)))\right.\right.$

$\left.\left.*(1/(b+(x[n])\hat{}(\mathrm{x2}[j,i]))\hat{}(a+n))\right],\{i,1,\mathrm{it1}\}\right];$

$\mathrm{Do}[\mathrm{If}[t[j,i]\leq(\mathrm{Log}[\mathrm{fx2}[j,i]/\mathrm{fmode}]),\{\mathrm{rv}[j,i]=\mathrm{x2}[j,i]\},\{\mathrm{rv}[j,i]=222222\}],\{i,1,\mathrm{it1}\}];$

$\mathrm{Do}[\mathrm{If}[\mathrm{rv}[j,k]\neq 222222,\{\mathrm{alpha}[j]=\mathrm{rv}[j,k],\mathrm{Break}[]\},\mathrm{alpha}[j]=\mathrm{rv}[j,k+1]],\{k,1,\mathrm{it1}\}];$

(*.....Generatingλfromπ1(λ/data)..........................*)

$\mu=a+n;\sigma=b+(x[n]\hat{}\mathrm{alpha}[j]);$

$\mathrm{dist}=\mathrm{GammaDistribution}[\mu,(1/\sigma)];$

$\mathrm{lambda}[j]=\mathrm{Random}[\mathrm{dist}];$

(*...*)

]; (* End of j loop *)

(*...ComputationsofiterationBayesestimators................*)

$\mathrm{alphaBE11}[l]=\left(\sum_{j=1}^{\mathrm{it}}\mathrm{alpha}[j]\right)\Big/\mathrm{it};$

$\mathrm{MSE31}[l]=\left(\sum_{j=1}^{\mathrm{it}}(\mathrm{alpha}[j]-\mathrm{alphaBE11}[l])\hat{}2\right)\Big/\mathrm{it};$

$\mathrm{lambdaBE11}[l]=\left(\sum_{j=1}^{\mathrm{it}}\mathrm{lambda}[j]\right)\Big/\mathrm{it};$

$\mathrm{MSE41}[l]=\left(\sum_{j=1}^{\mathrm{it}}(\mathrm{lambda}[j]-\mathrm{lambdaBE11}[l])\hat{}2\right)\Big/\mathrm{it};$

$\mathrm{data}\alpha=\mathrm{Table}[\mathrm{alpha}[j],\{j,1,\mathrm{it}\}];$

$\mathrm{order}\alpha=\mathrm{Sort}[\mathrm{data}\alpha];$

$\mathrm{data}\lambda=\mathrm{Table}[\mathrm{lambda}[j],\{j,1,\mathrm{it}\}];$

$\mathrm{order}\lambda=\mathrm{Sort}[\mathrm{data}\lambda];$

$\mathrm{alphaBE21}[l]=(\mathrm{order}\alpha[[\mathrm{it}/2]]+\mathrm{order}\alpha[[(\mathrm{it}/2)+1]])/2;$

$\mathrm{MSE51}[l]=\left(\sum_{j=1}^{\mathrm{it}}(\mathrm{alpha}[j]-\mathrm{alphaBE21}[l])\hat{}2\right)\Big/\mathrm{it};$

$\mathrm{lambdaBE21}[l]=(\mathrm{order}\lambda[[\mathrm{it}/2]]+\mathrm{order}\lambda[[(\mathrm{it}/2)+1]])/2;$

$\mathrm{MSE61}[l]=\left(\sum_{j=1}^{\mathrm{it}}(\mathrm{lambda}[j]-\mathrm{lambdaBE21}[l])\hat{}2\right)\Big/\mathrm{it};$

$\mathrm{alphaBE311}[l]=\left((1/\mathrm{it})*\sum_{j=1}^{\mathrm{it}}(1/(\mathrm{alpha}[j])\hat{}\mathrm{astar1})\right)\hat{}(-1/\mathrm{astar1});$

$\mathrm{MSE71}[l]=\left(\sum_{j=1}^{\mathrm{it}}(\mathrm{alpha}[j]-\mathrm{alphaBE311}[l])\hat{}2\right)\Big/\mathrm{it};$

$\mathrm{lambdaBE311}[l]=\left((1/\mathrm{it})*\sum_{j=1}^{\mathrm{it}}(1/(\mathrm{lambda}[j])\hat{}\mathrm{astar1})\right)\hat{}(-1/\mathrm{astar1});$

$\mathrm{MSE81}[l]=\left(\sum_{j=1}^{\mathrm{it}}(\mathrm{lambda}[j]-\mathrm{lambdaBE311}[l])\hat{}2\right)\Big/\mathrm{it};$

$\mathrm{alphaBE321}[l]=\left((1/\mathrm{it})*\sum_{j=1}^{\mathrm{it}}(1/(\mathrm{alpha}[j])\hat{}\mathrm{astar2})\right)\hat{}(-1/\mathrm{astar2});$

$\mathrm{MSE91}[l]=\left(\sum_{j=1}^{\mathrm{it}}(\mathrm{alpha}[j]-\mathrm{alphaBE321}[l])\hat{}2\right)\Big/\mathrm{it};$

$\mathrm{lambdaBE321}[l]=\left((1/\mathrm{it})*\sum_{j=1}^{\mathrm{it}}(1/(\mathrm{lambda}[j])\hat{}\mathrm{astar2})\right)\hat{}(-1/\mathrm{astar2});$

$\mathrm{MSE101}[l]=\left(\sum_{j=1}^{\mathrm{it}}(\mathrm{lambda}[j]-\mathrm{lambdaBE321}[l])\hat{}2\right)\Big/\mathrm{it};$

$\mathrm{alphaBE331}[l]=\left((1/\mathrm{it})*\sum_{j=1}^{\mathrm{it}}(1/(\mathrm{alpha}[j])\hat{}\mathrm{astar3})\right)\hat{}(-1/\mathrm{astar3});$

$\mathrm{MSE111}[l]=\left(\sum_{j=1}^{\mathrm{it}}(\mathrm{alpha}[j]-\mathrm{alphaBE331}[l])\hat{}2\right)\Big/\mathrm{it};$

$\mathrm{lambdaBE331}[l]=\left((1/\mathrm{it})*\sum_{j=1}^{\mathrm{it}}(1/(\mathrm{lambda}[j])\hat{}\mathrm{astar3})\right)\hat{}(-1/\mathrm{astar3});$

$\mathrm{MSE121}[l]=\left(\sum_{j=1}^{\mathrm{it}}(\mathrm{lambda}[j]-\mathrm{lambdaBE331}[l])\hat{}2\right)\Big/\mathrm{it};$

]; (*...Endof$l$loop....*)

(*...computationsofMLEsandBayesestimatorsandtheirMSEs....*)

(*..........MLEs...*)

alphaMLE = $\left(\sum_{l=1}^{M}\text{alphaMLE1}[l]\right)\Big/M$;

MSE1 = $\left(\sum_{l=1}^{M}(\text{alphaMLE1}[l]-\text{alphaMLE})\text{^}2\right)\Big/M$;

lambdaMLE = $\left(\sum_{l=1}^{M}\text{lambdaMLE1}[l]\right)\Big/M$;

MSE2 = $\left(\sum_{l=1}^{M}(\text{lambdaMLE1}[l]-\text{lambdaMLE})\text{^}2\right)\Big/M$;

(*....Bayesestimatorundersquareerrorlossfunction..........*)

alphaBE1 = $\left(\sum_{l=1}^{M}\text{alphaBE11}[l]\right)\Big/M$;

MSE3 = $\left(\sum_{l=1}^{M}\text{MSE31}[l]\right)\Big/M$;

lambdaBE1 = $\left(\sum_{l=1}^{M}\text{lambdaBE11}[l]\right)\Big/M$;

MSE4 = $\left(\sum_{l=1}^{M}\text{MSE41}[l]\right)\Big/M$;

(*.....Bayesestimatorunderabsoluteerrorlossfunction.........*)

alphaBE2 = $\left(\sum_{l=1}^{M}\text{alphaBE21}[l]\right)\Big/M$;

MSE5 = $\left(\sum_{l=1}^{M}\text{MSE51}[l]\right)\Big/M$;

lambdaBE2 = $\left(\sum_{l=1}^{M}\text{lambdaBE21}[l]\right)\Big/M$;

MSE6 = $\left(\sum_{l=1}^{M}\text{MSE61}[l]\right)\Big/M$;

(*......BayesestimatorunderLINRXlossfunction.........*)

(*.................. $(a^*=0.1)$ ....*)

alphaBE31 = $\left(\sum_{l=1}^{M}\text{alphaBE311}[l]\right)\Big/M$;

MSE7 = $\left(\sum_{l=1}^{M}\text{MSE71}[l]\right)\Big/M$;

lambdaBE31 = $\left(\sum_{l=1}^{M}\text{lambdaBE311}[l]\right)\Big/M$;

MSE8 = $\left(\sum_{l=1}^{M}\text{MSE81}[l]\right)\Big/M$;

(*.................. $(a^*=1)$ ....*)

alphaBE32 = $\left(\sum_{l=1}^{M}\text{alphaBE321}[l]\right)\Big/M$;

MSE9 = $\left(\sum_{l=1}^{M}\text{MSE91}[l]\right)\Big/M$;

lambdaBE32 = $\left(\sum_{l=1}^{M}\text{lambdaBE321}[l]\right)\Big/M$;

MSE10 = $\left(\sum_{l=1}^{M}\text{MSE101}[l]\right)\Big/M$;

(*.................. $(a^*=5)$ ....*)

alphaBE33 = $\left(\sum_{l=1}^{M}\text{alphaBE331}[l]\right)\Big/M$;

MSE11 = $\left(\sum_{l=1}^{M}\text{MSE111}[l]\right)\Big/M$;

lambdaBE33 = $\left(\sum_{l=1}^{M}\text{lambdaBE331}[l]\right)\Big/M$;

MSE12 = $\left(\sum_{l=1}^{M}\text{MSE121}[l]\right)\Big/M$;

(*...*)

```
Print["Prior 0 "];
Print["Case n = ", n];
Print ["Parameter | MLE | S.E.L | A.E.L | a*=", astar1, "| a*=", astar2, "| a*=", astar3, "|"] ;

Print[" α & ", NumberForm[Re[alphaMLE], 5], " & ", NumberForm[Re[alphaBE1], 5], " & ",
NumberForm[Re[alphaBE2], 5], " & ", NumberForm[Re[alphaBE31], 5], " & ",
NumberForm[Re[alphaBE32], 5], " & ", NumberForm[Re[alphaBE33], 5]];
Print[" MSE(α) & $(", NumberForm[Re[MSE1], 4], ")$&$(", NumberForm[Re[MSE3], 4],
")$&$(", NumberForm[Re[MSE5], 4], ")$&$(", NumberForm[Re[MSE7], 4], ")$&$(",
NumberForm[Re[MSE9], 4], ")$&$(", NumberForm[Re[MSE11], 4], ")$"];
Print[" λ & ", NumberForm[Re[lambdaMLE], 5], " & ", NumberForm[Re[lambdaBE1], 5], " & ",
NumberForm[Re[lambdaBE2], 5], " & ", NumberForm[Re[lambdaBE31], 5], " & ",
NumberForm[Re[lambdaBE32], 5], " & ", NumberForm[Re[lambdaBE33], 5]];
Print[" MSE(λ) & $(", NumberForm[Re[MSE2], 4], ")$&$(", NumberForm[Re[MSE4], 4],
")$&$(", NumberForm[Re[MSE6], 4], ")$&$(", NumberForm[Re[MSE8], 4], ")$&$(",
NumberForm[Re[MSE10], 4], ")$&$(", NumberForm[Re[MSE12], 4], ")$"];
```

Appendix 6

(* One sample prediction based on record data*)
$\alpha = 2; \lambda = 1;$ (* The assumed values of the weibull parameters*)
$a = 1; b = 1;$ (*Assumedvaluesofhyperparametersofπ1(λ/data)..Prior1....*)
$c = 1; d = 1;$ (*Assumedvaluesofhyperparametersofπ2(α)........Prior1....*)
it = 1000(* number of interations*);
$m = 9$(* size of record sample drawn from weibull *);
it1 = $m + 10$;
$\tau = 0.05$; (* Coverage of the predictive interval for the nth future record *)
astar31 = 0.1; astar32 = 0.5; astar33 = 1;
(*..genetatingrecorddatafromWeibull................................*)
Do[$u[i]$ = Random[], $\{i, 1, 100\}$];
$\beta 1 = 1$;
Do[$e[i] = (-1/\beta 1) *$ Log[$1 - u[i]$], $\{i, 1, 100\}$];
Do $\left[x[k] = \left(\left(\sum_{i=1}^{k}(e[i])\right)\Big/\lambda\right)\hat{}(1/\alpha), \{k, 1, m\}\right]$;
Print["The observed record sample is "];
Do[Print["x(u(", i, ")) = ", $x[i]$], $\{i, 1, m\}$];
mode =
a1/.
FindRoot $[-d + ((c + m - 1)/\text{a1}) + (\sum_{i=1}^{m} \text{Log}[x[i]]) - ((a + m) * (x[m]\hat{}\text{a1})$
$*\text{Log}[x[m]])/(b + x[m]\hat{}\text{a1}) == 0, \{\text{a1}, 2\}]$;
fmode = Log $\left[e^{-d*\text{mode}} * (\text{mode}\hat{}(c + m - 1)) * (\prod_{i=1}^{m}((x[i])\hat{}(\text{mode} - 1)))\right.$
$\left.*(1/(b + (x[m]\hat{}\text{mode}))\hat{}(a + m))\right]$;
For[$j = 1, j \leq$ it, j++, (* loop of iterations *)
$\theta = 1$;
Do[u2[j, i] = 2 * Random[], $\{i, 1, \text{it1}\}$];
Do[u3[j, i] = Random[], $\{i, 1, \text{it1}\}$];
Do[u4[j, i] = Random[], $\{i, 1, \text{it1}\}$];
Do[$e[j, i] = (-1/\theta) *$ Log[1 − u3[j, i]], $\{i, 1, \text{it1}\}$];
Do[e1[j, i] = $(-1/\theta) *$ Log[1 − u4[j, i]], $\{i, 1, \text{it1}\}$];
Do[If[u2[j, i] ≤ 1, {xx[j, i] = u2[j, i], $t[j, i] = -e[j, i]$}, {xx[j, i] = 1 + e1[j, i],
$t[j, i] = -e[j, i] -$ e1[j, i]}], $\{i, 1, \text{it1}\}$];
Do[x2[j, i] = mode + (xx[j, i]/fmode), $\{i, 1, \text{it1}\}$];
Do $\left[\text{fm}[j, i] = \text{Log}\left[e^{-d*\text{x2}[j,i]} * ((\text{x2}[j, i])\hat{}(c + m - 1)) * (\prod_{i=1}^{m}((x[i])\hat{}(\text{x2}[j, i] - 1)))\right.\right.$
$\left.\left.*(1/(b + (x[m])\hat{}(\text{x2}[j, i]))\hat{}(a + m))\right], \{i, 1, \text{it1}\}\right]$;
Do[kr[j, i] = (Log[fm[j, i]/fmode]), $\{i, 1, \text{it1}\}$];
Do[If[$t[j, i] \leq$ (Log[fm[j, i]/fmode]), {rv[j, i] = x2[j, i]}, {rv[j, i] = 222222}], $\{i, 1, \text{it1}\}$];
Do[If[rv[j, k] $\neq$ 222222, {alpha[j] = rv[j, k], Break[]}, alpha[j] = rv[$j, k + 1$]], $\{k, 1, \text{it1}\}$];
$\mu = a + m; \sigma = b + (x[m]\hat{}\text{alpha}[j])$;

$\text{dist} = \text{GammaDistribution}[\mu, (1/\sigma)];$
$\text{lambda}[j] = \text{Random}[\text{dist}];$
$]; (* \text{ End of loop of iterations } *)$
$\text{For}[n = m + 1, n \leq m + 5, n{+}{+},$
$\text{For}[j = 1, j \leq \text{it}, j{+}{+},$
$\text{sum1}[j] = ((\text{lambda}[j]\hat{}(n - m - 1 - (1/\text{alpha}[j])))*$
$e^{\text{lambda}[j]*(x[m]\hat{}\text{alpha}[j])} / ((n - m - 1)!)) * \left(\sum_{i=0}^{n-m-1}(\text{Binomial}[n - m - 1, i]\right.$
$*((-1)\hat{}(n - m - i - 1)) * (x[m]\hat{}((n - m - i - 1) * \text{alpha}[j]))$
$*(\text{Gamma}[(1/\text{alpha}[j]) + i + 1, \text{lambda}[j]$
$*(x[m]\hat{}\text{alpha}[j])]/(\text{lambda}[j]\hat{}i))));$
$\text{Survival}[j] = \sum_{i=0}^{n-m-1} \left((e^{-\text{lambda}[j]*(y\hat{}\text{alpha}[j]-x[m]\hat{}\text{alpha}[j])}\right.$
$*((\text{lambda}[j] * (y\hat{}\text{alpha}[j] - x[m]\hat{}\text{alpha}[j]))\hat{}i))/(i!));$
$\text{sum2}[j] = \left(e^{\text{lambda}[j]*(x[m]\hat{}\text{alpha}[j])} / ((n - m - 1)!)\right) * \left(\sum_{i=0}^{n-m-1}(\text{Binomial}[n - m - 1, i]*\right.$
$((-\text{lambda}[j])\hat{}(n - m - i - 1)) * (x[m]\hat{}((n - m - i - 1)$
$*\text{alpha}[j])) * \text{Gamma}[i + 1, \text{lambda}[j] * (w\hat{}\text{alpha}[j])]));$
$\text{sum31}[j] = (((\text{lambda}[j]\hat{}(n - m - 1 + (\text{astar31}/\text{alpha}[j])))$
$* e^{\text{lambda}[j]*(x[m]\hat{}\text{alpha}[j])} / ((n - m - 1)!)) * \sum_{i=0}^{n-m-1}(\text{Binomial}[n - m - 1, i]$
$*((-1)\hat{}(n - m - i - 1)) * (x[m]\hat{}((n - m - i - 1) * \text{alpha}[j]))$
$*(\text{Gamma}[(-\text{astar31}/\text{alpha}[j]) + i + 1, \text{lambda}[j]$
$*(x[m]\hat{}\text{alpha}[j])]/(\text{lambda}[j]\hat{}i))));$
$\text{sum32}[j] = (((\text{lambda}[j]\hat{}(n - m - 1 + (\text{astar32}/\text{alpha}[j])))$
$* e^{\text{lambda}[j]*(x[m]\hat{}\text{alpha}[j])} / ((n - m - 1)!)) *$
$\sum_{i=0}^{n-m-1}(\text{Binomial}[n - m - 1, i] * ((-1)\hat{}(n - m - i - 1)) * (x[m]$
$\hat{}((n - m - i - 1) * \text{alpha}[j])) * (\text{Gamma}[(-\text{astar32}/\text{alpha}[j]) + i + 1,$
$\text{lambda}[j] * (x[m]\hat{}\text{alpha}[j])]/(\text{lambda}[j]\hat{}i))));$
$\text{sum33}[j] = (((\text{lambda}[j]\hat{}(n - m - 1 + (\text{astar33}/\text{alpha}[j])))$
$* e^{\text{lambda}[j]*(x[m]\hat{}\text{alpha}[j])} / ((n - m - 1)!)) * \sum_{i=0}^{n-m-1}(\text{Binomial}[n - m - 1, i]$
$*((-1)\hat{}(n - m - i - 1)) * (x[m]\hat{}((n - m - i - 1) * \text{alpha}[j]))$
$*(\text{Gamma}[(-\text{astar33}/\text{alpha}[j]) + i + 1,$
$\text{lambda}[j] * (x[m]\hat{}\text{alpha}[j])]/(\text{lambda}[j]\hat{}i))));$
$];$
$\text{Linitial} = x[m] + 0.3;$
$\text{Uinitial} = \text{Linitial} + 0.5;$
$(*......\text{Undersquareerrorlossfunction}...............*)$
$$\text{yBP1} = \left(\sum_{j=1}^{\text{it}} \text{sum1}[j]\right) \Big/ \text{it};$$
$$\text{PSurvival1} = \left(\sum_{j=1}^{\text{it}} \text{Survival}[j]\right) \Big/ \text{it};$$
$\text{Lower1} =$

$y/.$
FindRoot[PSurvival1 == $(1-(\tau/2)), \{y, \text{Linitial}\}$];
Upper1 =
$y/.$
FindRoot[PSurvival1 == $(\tau/2), \{y, \text{Uinitial}\}$];
(*... Underabsoluteerrorlossfunction.............*)
$$\text{sum} = \left(\sum_{j=1}^{\text{it}} \text{sum2}[j]\right) \Big/ \text{it};$$
yBP2 =
$w/.$
FindRoot[sum == 0.5, $\{w, \text{Linitial}\}$];

Survivals = Table[Survival[j], $\{j, 1, \text{it}\}$];

sort = Sort[Survivals];

PSurvival2 = (sort[[it/2]] + sort[[(it/2) + 1]])/2;
Lower2 =
$y/.$
FindRoot[PSurvival2 == $(1-(\tau/2)), \{y, \text{Linitial}\}$];
Upper2 =
$y/.$
FindRoot[PSurvival2 == $(\tau/2), \{y, \text{Uinitial}\}$];
(*.... UnderLINEXlossfunction ....*)
$$\text{yBP31} = \left(\left(\sum_{j=1}^{\text{it}} \text{sum31}[j]\right) \Big/ \text{it}\right)\hat{}(-1/\text{astar31});$$
$$\text{PSurvival31} = \left(\left(\sum_{j=1}^{\text{it}} \text{Survival}[j]\hat{}(-\text{astar31})\right) \Big/ \text{it}\right)\hat{}(-1/\text{astar31});$$
Lower31 =
$y/.$
FindRoot[PSurvival31 == $(1-(\tau/2)), \{y, \text{Linitial}\}$];
Upper31 =
$y/.$
FindRoot[PSurvival31 == $(\tau/2), \{y, \text{Uinitial}\}$];

$$\text{yBP32} = \left(\left(\sum_{j=1}^{\text{it}} \text{sum32}[j]\right) \Big/ \text{it}\right)\hat{}(-1/\text{astar32});$$
$$\text{PSurvival32} = \left(\left(\sum_{j=1}^{\text{it}} \text{Survival}[j]\hat{}(-\text{astar32})\right) \Big/ \text{it}\right)\hat{}(-1/\text{astar32});$$
Lower32 =

y/.
FindRoot[PSurvival32 == (1 − (τ/2)), {y, Linitial}];

Upper32 =
y/.
FindRoot[PSurvival32 == (τ/2), {y, Uinitial}];

$$\text{yBP33} = \left(\left(\sum_{j=1}^{\text{it}} \text{sum33}[j]\right) \Big/ \text{it}\right) \text{\^{}}(-1/\text{astar33});$$

$$\text{PSurvival33} = \left(\left(\sum_{j=1}^{\text{it}} \text{Survival}[j]\text{\^{}}(-\text{astar33})\right) \Big/ \text{it}\right) \text{\^{}}(-1/\text{astar33});$$

Lower33 =
y/.
FindRoot[PSurvival33 == (1 − (τ/2)), {y, Linitial}];
Upper33 =
y/.
FindRoot[PSurvival33 == (τ/2), {y, Uinitial}];

```
Print["\\multirow{1}{*}{}&$ X_{(U(", n, "))}", " $&", "Sq. err. &",
NumberForm[Re[yBP1], 5], " &$(", NumberForm[Re[Lower1], 5], ",",
NumberForm[Re[Upper1], 5], ")$\\\\"];
Print["\\multirow{1}{*}{}&&", "Ab. err. &",
NumberForm[Re[yBP2], 5], " &$(", NumberForm[Re[Lower2], 5], ",",
NumberForm[Re[Upper2], 5], ")$\\\\"];
Print ["\\multirow{1}{*}{}&&", "LINEX $(a*=0.1)$ &",
NumberForm[Re[yBP31], 5], " &$(", NumberForm[Re[Lower31], 5], ",",
NumberForm[Re[Upper31], 5], ")$\\\\"];
Print ["\\multirow{1}{*}{}&&", "LINEX $(a*=0.5)$ &",
NumberForm[Re[yBP32], 5], " &$(", NumberForm[Re[Lower32], 5], ",",
NumberForm[Re[Upper32], 5], ")$\\\\"];
Print ["\\multirow{1}{*}{}&&", "LINEX $(a*=1.0)$ &",
NumberForm[Re[yBP33], 5], " &$(", NumberForm[Re[Lower33], 5], ",",
NumberForm[Re[Upper33], 5], ")$\\\\ \cline{2-6}"];
];
```

Appendix 7

(*ComputatiosoftwosamplesBayespredictorsunderdifferentlossfunctions,
andpredictionintervals, basedonrecorddata, byusingprior1*)
$\alpha = 2; \lambda = 1;$ (* The assumed values of the weibull parameters*)
$a = 1; b = 1;$ (*Assumedvaluesofhyperparametersof$\pi 1(\lambda/\text{data})$..Prior1..*)
$c = 2; d = 1;$ (*Assumedvaluesofhyperparametersof$\pi 2(\alpha)$.......Prior1...*)
it = 1000(* number of interations*);
$m = 9;$ (*no.ofrecords*)
$n = 6;$ (* size of future record sample *)
$\text{it1} = m + 5;$
$\tau = 0.05;$ (* Coverage of the predictive interval for the nth future record *)
astar1 = 0.1; astar2 = 0.5; astar3 = 1;
$\text{Do}[u[i] = \text{Random}[], \{i, 1, 100\}];$
$\beta 1 = 1;$
$\text{Do}[e[i] = (-1/\beta 1) * \text{Log}[1 - u[i]], \{i, 1, 100\}];$
$\text{Do}\left[x[k] = \left(\left(\sum_{i=1}^{k}(e[i])\right) \Big/ \lambda\right) \hat{}(1/\alpha), \{k, 1, m\}\right];$
Print["The observed recored sample is of size m = ", m];
Print["The observed record sample is "];
Do[Print["x(u(", i, ")) = ", $x[i]$], $\{i, 1, m\}$];
mode =
a1/.
$\text{FindRoot}\,[-d + ((c + m - 1)/\text{a1}) + (\sum_{i=1}^{m} \text{Log}[x[i]]) - ((a + m) * (x[m]\hat{}\text{a1})$
$*\text{Log}[x[m]])/(b + x[m]\hat{}\text{a1}) == 0, \{\text{a1}, 2\}];$
$\text{fmode} = \text{Log}\left[e^{-d*\text{mode}} * (\text{mode}\hat{}(c + m - 1)) * (\prod_{i=1}^{m}((x[i])\hat{}(\text{mode} - 1)))\right.$
$\left.*(1/(b + (x[m]\hat{}\text{mode}))\hat{}(a + m))\right];$
For[$j = 1, j \leq \text{it}, j{+}{+}$, (* loop of iterations *)
(*.Generating$\alpha$from$\pi 2(\alpha/\text{data})$byusingDevroye(1984)method........*)
$\theta = 1;$
$\text{Do}[\text{u2}[j, i] = 2 * \text{Random}[], \{i, 1, \text{it1}\}];$
$\text{Do}[\text{u3}[j, i] = \text{Random}[], \{i, 1, \text{it1}\}];$
$\text{Do}[\text{u4}[j, i] = \text{Random}[], \{i, 1, \text{it1}\}];$
$\text{Do}[e[j, i] = (-1/\theta) * \text{Log}[1 - \text{u3}[j, i]], \{i, 1, \text{it1}\}];$
$\text{Do}[\text{e1}[j, i] = (-1/\theta) * \text{Log}[1 - \text{u4}[j, i]], \{i, 1, \text{it1}\}];$
$\text{Do}[\text{If}[\text{u2}[j, i] \leq 1, \{\text{xx}[j, i] = \text{u2}[j, i], t[j, i] = -e[j, i]\}, \{\text{xx}[j, i] = 1 + \text{e1}[j, i],$
$t[j, i] = -e[j, i] - \text{e1}[j, i]\}], \{i, 1, \text{it1}\}];$
$\text{Do}[\text{x2}[j, i] = \text{mode} + (\text{xx}[j, i]/\text{fmode}), \{i, 1, \text{it1}\}];$
$\text{Do}\left[\text{fx2}[j, i] = \text{Log}\left[e^{-d*\text{x2}[j,i]} * ((\text{x2}[j, i])\hat{}(c + m - 1)) * (\prod_{i=1}^{m}((x[i])\hat{}(\text{x2}[j, i] - 1)))\right.\right.$
$\left.\left.*(1/(b + (x[m])\hat{}(\text{x2}[j, i]))\hat{}(a + m))\right], \{i, 1, \text{it1}\}\right];$
$\text{Do}[\text{If}[t[j, i] \leq (\text{Log}[\text{fx2}[j, i]/\text{fmode}]), \{\text{rv}[j, i] = \text{x2}[j, i]\}, \{\text{rv}[j, i] = 222222\}], \{i, 1, \text{it1}\}];$

Do[If[rv[j, k] $\neq$ 222222, {alpha[j] = rv[j, k], Break[]}, alpha[j] = rv[$j, k + 1$]], {k, 1, it1}];
(*...Generatingλfromπ1(λ/data)........................*)
$\mu = m + a; \sigma = b +$ (x[m]^alpha[j]);
dist = GammaDistribution[μ, (1/σ)];
lambda[j] = Random[dist];
(*..Endofgeneration.....................................*)
]; (* End of j loop *)
Linitial = x[1];
Uinitial = Linitial + 1;
Minitial = x[1];
For[$k = 1, k \leq n, k$++,
For[$j = 1, j \leq$ it, j++,
(*Computationofdensity, distributionfunctionandsurvivalfunction*)
sum1[j] = (Gamma[k + (1/alpha[j])]/((lambda[j]^(1/alpha[j])) $*$ Gamma[k]));
sum2[j] = (Gamma[k, lambda[j] $*$ (y^alpha[j])]/Gamma[k]);
sum31[j] = (((lambda[j]^(astar1/alpha[j]))
$*$ Gamma[k − (astar1/alpha[j])]])/Gamma[k]);
sum32[j] = (((lambda[j]^(astar2/alpha[j]))
$*$ Gamma[k − (astar2/alpha[j])]])/Gamma[k]);
sum33[j] = (((lambda[j]^(astar3/alpha[j]))
$*$ Gamma[k − (astar3/alpha[j])]])/Gamma[k]);
sum6[j] = 1 − (Gamma[k, (lambda[j] $*$ (y^alpha[j]))]/Gamma[k]);
];
(*.....squareerrorlossfunction....................*)
yBP1 = $\left(\sum_{j=1}^{\text{it}} \text{sum1}[j]\right)\Big/$ it;
PDF1 = $\left(\sum_{j=1}^{\text{it}} \text{sum6}[j]\right)\Big/$ it;
Lower1 =
y/.
FindRoot[PDF1 == (τ/2), {y, Linitial}];
Upper1 =
y/.
FindRoot[PDF1 == (1 − (τ/2)), {y, Uinitial}];
(*....absoluteerrorlossfunction................*)
sum = $\left(\sum_{j=1}^{\text{it}} \text{sum2}[j]\right)\Big/$ it;
yBP2 =
y/.
FindRoot[sum == 0.5, {y, Minitial}];
DFs = Table[sum6[j], {j, 1, it}];
sort = Sort[DFs];

```
PDF2 = (sort[[it/2]] + sort[[(it/2) + 1]])/2;
Lower2 =
y/.
FindRoot[PDF2 == (τ/2), {y, Linitial}];
Upper2 =
y/.
FindRoot[PDF2 == (1 − (τ/2)), {y, Uinitial}];
(*....LINEXlossfunction......................*)
(*...........a* = 0.1............................*)
yBP31 = ((Sum[sum31[j], {j, 1, it}]) / it)^(−1/astar1);
PDF31 = ((Sum[(sum6[j]^(−astar1)), {j, 1, it}]) / it)^(−1/astar1);
Lower31 =
y/.
FindRoot[PDF31 == (τ/2), {y, Linitial}];
Upper31 =
y/.
FindRoot[PDF31 == (1 − (τ/2)), {y, Uinitial}];
(*........a* = 0.5.............................*)
yBP32 = ((Sum[sum32[j], {j, 1, it}]) / it)^(−1/astar2);

PDF32 = ((Sum[(sum6[j]^(−astar2)), {j, 1, it}]) / it)^(−1/astar2);
Lower32 =
y/.
FindRoot[PDF32 == (τ/2), {y, Linitial}];
Upper32 =
y/.
FindRoot[PDF32 == (1 − (τ/2)), {y, Uinitial}];
(*.......a* = 1...............................*)
yBP33 = ((Sum[sum33[j], {j, 1, it}]) / it)^(−1/astar3);
PDF33 = ((Sum[(sum6[j]^(−astar3)), {j, 1, it}]) / it)^(−1/astar3);
Lower33 =
y/.
FindRoot[PDF33 == (τ/2), {y, Linitial}];
Upper33 =
y/.
FindRoot[PDF33 == (1 − (τ/2)), {y, Uinitial}];
(*...................................................................*)
Print["\\multirow{1}{*}{}&&$ Y_{(U(", k, "))}", " $&", "Sq. err. &",
```

```
NumberForm[Re[yBP1], 5], " &$(", NumberForm[Re[Lower1], 5], ",",
NumberForm[Re[Upper1], 5], ")$\\\\"];
Print["\\multirow{1}{*}{}&&&", "Ab. err. &",
NumberForm[Re[yBP2], 5], " &$(", NumberForm[Re[Lower2], 5], ",",
NumberForm[Re[Upper2], 5], ")$\\\\"];
Print ["\\multirow{1}{*}{}&&&", "LINEX $(a*=0.1)$ &",
NumberForm[Re[yBP31], 5], " &$(", NumberForm[Re[Lower31], 5], ",",
NumberForm[Re[Upper31], 5], ")$\\\\"];
Print ["\\multirow{1}{*}{}&&&", "LINEX $(a*=0.5)$ &",
NumberForm[Re[yBP32], 5], " &$(", NumberForm[Re[Lower32], 5], ",",
NumberForm[Re[Upper32], 5], ")$\\\\"];
Print ["\\multirow{1}{*}{}&&&", "LINEX $(a*=1.0)$ &",
NumberForm[Re[yBP33], 5], " &$(", NumberForm[Re[Lower33], 5], ",",
NumberForm[Re[Upper33], 5], ")$\\\\ \cline{3-6}"];
Linitial = Lower1 + 0.5;
Uinitial = Upper1 + 0.5;
Minitial = yBP2 + 0.5;
];
```

الإستدلال الإحصائي و التنبؤ بوجود الإحصائيات الرتبية

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ملخــــــص

في هذا العمل البحثي، قدرنا معلمتي توزيع وايبل باستخدام طريقة تعظيم اقتران الاحتمالية و طريقة بييز، و كذلك تنبئنا عن قيم مفقودة و مستقبلية بالأعتماد على عينات تمت مشاهدتها من نوع بيانات الرقابة التقدمية من النوع الثاني و البيانات المسجلة الرتبية و المأخوذة من توزيع وايبل. هذه التنبئات تم ايجادها بطريقتين الاولى هي طريقة العينة الواحدة للتنبؤ و الثانية هي طريقة العينتين للتنبؤ. يلاحظ من المعادلات المشتقة و الممثلة لهذه التقديرات و التنبؤات انها تحتوي على تكاملات ثنائية لا يمكن اجرائها، هنا طورنا طريقة لمحاكاة الكمبيوتر لسحب عينات من التوزيع البعدي لمعلمتي توزيع وايبل باستخدام طريقة قبس و استخدام هذه العينات المولدة لتقريب هذه التقديرات و التنبؤات. تم عرض نتائج بالاعتماد على بيانات مولدة بطريقة محاكاة الكمبيوتر و بيانات حقيقية لرؤية فعالية هذه الطرق.